MATH 5620 – NUMERICAL ANALYSIS II
PRACTICE MIDTERM EXAM

Problem 1. Consider the multistep method:
\[ y_n - y_{n-2} = \frac{h}{3}[f_n - 3f_{n-1} + 2f_{n-2}] \]

(a) Is this method implicit or explicit?
(b) Is this method convergent, stable and/or consistent? Justify your answer.

Problem 2. The goal of this problem is to design a non-linear shooting method for solving a non-linear BVP with mixed type boundary conditions:
\[ \begin{cases} 
  y'' = f(t, y, y'), & \text{for } t \in [a, b], \\
  y'(a) = \alpha, \\
  y(b) = \beta. 
\end{cases} \quad (1) \]

Let \( y = y(t; z) \) be the solution to the IVP
\[ \begin{cases} 
  y'' = f(t, y, y'), & \text{for } t \in [a, b], \\
  y(a) = z, \\
  y'(a) = \alpha. 
\end{cases} \quad (2) \]

and let \( \phi(z) = y(b; z) - \beta. \)

(a) Let \( u(t) = \frac{\partial u(t; z)}{\partial z}. \) By differentiating (2) with respect to \( z, \) show that \( u \) solves
\[ \begin{cases} 
  u'' = u f_y(t, y, y') + u' f_{y'}(t, y, y'), & \text{for } t \in [a, b], \\
  u(a) = 1, \\
  u'(a) = 0. 
\end{cases} \]

(b) Show that \( \phi'(z) = u(b). \)
(c) Assuming the availability of a routine for solving systems of first order equations, write pseudocode for solving (1), based on Newton’s method for finding \( z \) such that \( \phi(z) = 0. \)

Problem 3.
(a) Using the method of undetermined coefficients derive the second order Adams-Moulton formula of the form
\[ y_{n+1} = y_n + h[A f_{n+1} + B f_n] \]
(b) Recall that for a linear multistep method of the form
\[ a_k y_n + a_{k-1} y_{n-1} + \cdots + a_0 y_{n-k} = h \left[ b_k f_n + b_{k-1} f_{n-1} + \cdots + b_0 f_{n-k} \right] \]
we associate the linear functional
\[ L_y = \sum_{i=0}^{k} [a_i y(ih) - h b_i y'(ih)] \]
which has Taylor expansion at $t = 0$: 

$$Ly = d_0 y(0) + d_1 y'(0) + d_2 h^2 y''(0) + \cdots$$

where

$$d_0 = \sum_{i=0}^{k} a_k$$

$$d_j = \sum_{i=0}^{k} \left( \frac{i^j}{j!} a_i - \frac{i^{j-1}}{(j-1)!} b_i \right), \quad j \geq 1.$$  

Verify that this is an order 2 method and find the local truncation error, i.e. find the constant $C$ for which

$$y(t_n) - y_n = Ch^3 y^{(3)}(t_{n-1}) + O(h^4),$$

where the previous values $y_{n-1}, y_{n-2}, \ldots$ are assumed exact.

**Problem 4.** Consider the IVP

$$\begin{cases}
y' = f(t, y) \\
y(a) = \alpha.
\end{cases}$$

(a) Write the first 3 terms of the Taylor series for $y(t + h)$ expanding around $t$. Your series should be in terms of $h, f$ and its partial derivatives. The residual should be $O(h^3)$.

(b) Recall the general formula for a second-order Runge-Kutta method:

$$y(t + h) = y + w_1 hf + w_2 hf(t + \alpha h, y + \beta hf) + O(h^3),$$

where $y \equiv y(t)$ and $f \equiv f(t, y)$. Use the two variable Taylor expansion

$$f(t + hs, y + hv) = f(t, y) + hs f_t(t, y) + hv f_y(t, y) + O(h^2)$$

to express (4) in terms of $y, f, f_t \equiv f_t(t, y)$ and $f_y \equiv f_y(t, y)$.

(c) What conditions should $w_1, w_2, \alpha$ and $\beta$ satisfy in order for the method to be second order?

(d) Write down the particular Runge-Kutta method of order 2 with $w_1 = 1/4$. 