Hyperbolic Equations

Used to model wave motion: acoustics, seismics, electromagnetics...
Also model "advection": transport of a substance, this is simpler to deal with so we will stick to advection to illustrate the different numerical methods for Hyperbolic Equations.

Advection:

$$\begin{cases} u_t + a u_x = 0 & \text{, where } a = \text{constant} \\ u(x,0) = \eta(x) \end{cases}$$

exact solution is $u(x,t) = \eta(x-at)$ (check)

The first approach that comes to mind is to discretize:

$$u_x(x,t) = \frac{u(x+h,t) - u(x-h,t)}{2h} + O(h^2)$$

= central differences in space

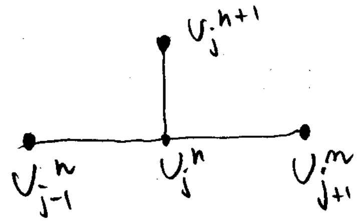
$$u_t(x,t) = \frac{u(x,t+k) - u(x,t)}{k} + O(k)$$

= forward differences in time

we get the scheme:

$$\frac{U_j^{n+1} - U_j^n}{k} = -\frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

or $$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$



(stencil)

not practical because of stability reasons.

Lax Friedrichs method:

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

(not used too much in practice because of low accuracy)

This is a Lax-Richtmyer stable method and convergent provided

$$\left|\frac{ak}{h}\right| \leq 1 \qquad (*)$$

i.e. $k = O(h)$. This is rather large compared to the $O(h^2)$ time step we had to take for Heat eq.

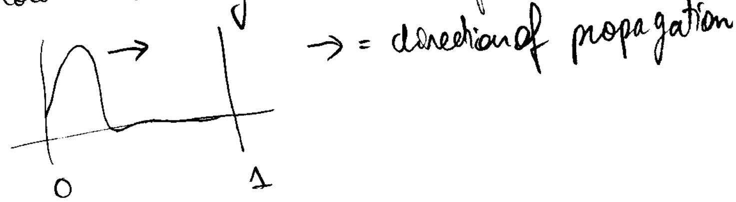
(*) makes a lot of sense and is actually called the Courant-Friedrichs-Lewy condition (CFL):

Recall
$$\left.\begin{aligned} u(x,t) &= \eta(x-at) \\ u_x(x,t) &= \eta'(x-at) \\ u_t(x,t) &= -a\,\eta'(x-at) \end{aligned}\right\} \Rightarrow u_t = a\,u_x$$

we need temporal resolution to be a times smaller than spatial resolution because solution changes in time a times faster than in space.

Method of lines Consider the problem

$$\begin{cases} u_t + a u_x = 0 \\ u(0,t) = g_0(t) \end{cases}$$ "inflow" boundary condition (if $a>0$)



This is more adapted to bounded domains than the Cauchy problem (initial data).

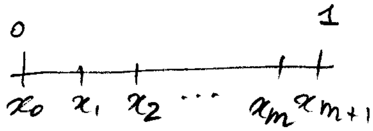
Note: if $a<0$ then inflow B.C. would be at 1 because direction of propagation would be reversed.

However stability is still easier to analyze if we assume periodic boundary conditions:

$$u(0,t) = u(1,t)$$

i.e. inflow at one end = outflow at other end.

(happens to agree with Cauchy problem if Cauchy data is periodic)



Let $U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_{m+1}(t) \end{bmatrix}$ where $U_j(t) \approx u(x_j, t)$

Using centered finite differences:

$$U_j'(t) = -\frac{a}{2h}\left(U_{j+1}(t) - U_{j-1}(t)\right) \qquad j = 2 \ldots m$$

$$U_1'(t) = -\frac{a}{2h}\left(U_2(t) - U_{m+1}(t)\right) \qquad j = 1$$

$$U_{m+1}'(t) = -\frac{a}{2h}\left(U_1(t) - U_m(t)\right) \qquad j = m+1$$

Or in system form:

$$U'(t) = AU(t)$$

with $A = -\dfrac{a}{2h} \begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ 1 & & & -1 & 0 \end{bmatrix} \in \mathbb{R}^{m+1 \times m+1}$

eigenvalues of A are purely imaginary $(A^T = -A)$:

$$\lambda_p = -\frac{i a}{h}\sin(2\pi p h), \text{ with eigenvectors } u^p:$$

$$u_j^p = e^{2i\pi p j h} \qquad p = 1, 2, \ldots, m+1$$

$$j = 1, 2, \ldots, m+1$$

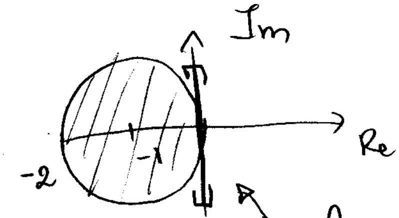
Concretely eigenvalues lie in an interval $-\frac{ia}{h}, \frac{ia}{h}$ in imaginary axis, so we need absolute stability region to include this.

<u>Forward Euler</u>: (first method we saw):

$$U_j^{n+1} = U_j^n + kA$$

$$\frac{U_j^{n+1} - U_j^n}{k} = -\frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

Abs stability region: $|1 + k\lambda| \leq 1$



eigenvalues of A always fall outside abs. stab. reg.

$\Rightarrow$ method is always unstable because eigenvalues $k\lambda$ do not lie in abs stab region if we let $\frac{k}{h}$ constant (fixed).

However if we let $k \to 0$ faster than h then $k\lambda_p \to 0$ so we get convergence (but not very practical if we take very small time steps!)

In terms of Lax Richtmyer stability:

$$B = 1 + kA$$

$$|\underbrace{1}_{\text{real}} + \underbrace{k\lambda_p}_{\text{imag}}|^2 = 1 + |k\lambda_p|^2 \leq 1 + \left(\frac{ka}{h}\right)^2$$

$$\leq 1 + a^2 h^2 = 1 + a^2 k$$

using e.g. $k = h^2$

$$\Rightarrow \|I + kA\|_2^2 \leq 1 + a^2 k$$

if $nk \leq T$

$$\|(I + kA)^n\|_2 \leq (1 + a^2 k)^{\frac{n}{2}} \leq e^{a^2 T/2}$$

$\Rightarrow \|B^n\|$ is uniformly bounded.

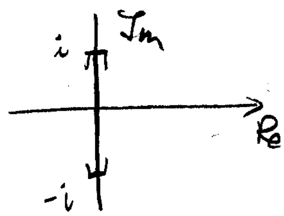
Leapfrog: use the mid point method for time discr:

$$\frac{U^{n+1} - U^{n-1}}{2k} = f(U^n) = AU^n = \text{centered fin. diff in space}$$

$$\Rightarrow U_j^{n+1} = U_j^{n-1} - \frac{ak}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

this is a 3 step method which is explicit, and second order in time and space.

Absolute stability region: is segment $[-1,1]$ on imaginary axis



$\Rightarrow$ need $-1 \le \mathrm{Im}\, k\lambda_p \le 1$

$\Rightarrow$ need $\left|\frac{ak}{h}\right| < 1$

Here $k\lambda_p$ is always right on the boundary of the absolute stability region

- all modes in the solution are preserved (no decay nor growth) (another way of saying this is that the method is non dissipative)
- however some modes may propagate at different speeds (numerical dispersion)

Leapfrog is not suitable to variable coefficient or non-linear problems because of this "marginal stability". (easy to fall off absolute stability region)

Lax-Friedrichs

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

We rewrite the first term as follows:

$$\frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) = U_j^n + \frac{1}{2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right)$$

Therefore:

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{1}{2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right)$$

Or to reveal more structure we can rewrite L.F. method as follows:

$$\frac{U_j^{n+1} - U_j^n}{k} + a\left(\frac{U_{j+1}^n - U_{j-1}^n}{2h}\right) = \frac{h^2}{2k}\,\frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}$$

Looks like sol to

$$u_t + a u_x = \underbrace{\varepsilon u_{xx}}_{\text{dissipation built-in the method.}}, \quad \varepsilon = \frac{h^2}{2k}$$

method could be obtained from discretizing:

$$U'(t) = A_\varepsilon U(t)$$

where

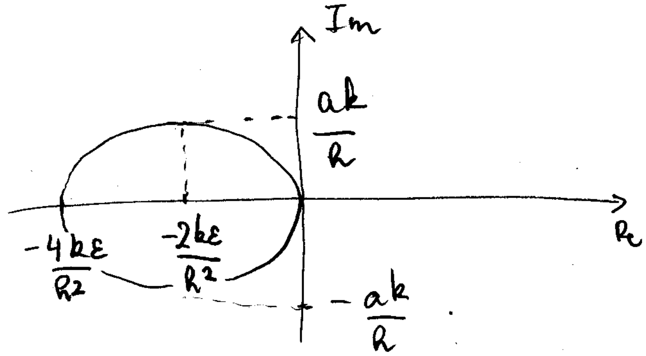
$$A_\varepsilon = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \\ 1 & & & & 1 \\ & & & -1 & 0 \end{bmatrix} + \frac{\varepsilon}{h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ 1 & & & & 1 \\ & & & 1 & -2 \end{bmatrix}$$

Since we are adding a small multiple of Laplacian and this matrix has same eigenvectors as A (the matrix that discretizes $-a u_x$), it follows that the eigenvalues of A will be shifted to the left in the complex plane.

Actually the eigenvalues are:

$$\lambda_p(A_\varepsilon) = \underbrace{\frac{-ia}{h}\sin(2\pi p h)}_{\text{difference op part}} - \underbrace{\frac{2\varepsilon}{h^2}\left(1-\cos(2\pi p h)\right)}_{\text{Laplacian part}}$$

Then if we plot $k\lambda$ in complex plane we obtain:



For $\varepsilon = \frac{h^2}{2k}$ that we use in Lax Friedrichs we have:

$$-\frac{2k\varepsilon}{h^2} = -1$$

Thus all eigenvalues fall inside area of absolute stability for Euler's method, of course provided that $\left|\frac{ak}{h}\right| \le 1$

Lax - Wendroff method · (see Carl's project as well)

Second order accurate in time and space (like Leapfrog)
but LW includes dissipation and works on one step only.

Idea is to discretize system of ODE's

$$U'(t) = A\,U(t)$$

that we get from Method of Lines.

$$U'' = AU' = A^2 U$$

$$\Rightarrow U^{n+1} = U^n + k\,U' + \frac{k^2}{2} U''$$

$$= U^n + kAU^n + \frac{k^2}{2} A^2 U^n$$

Computing A^2 gives scheme:

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{a^2k^2}{8h^2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right)$$

Upwind methods:

What if we wanted to use one sided differences?

$$u_x(x_j, t) \approx \frac{1}{h}\left(U_j - U_{j-1}\right)$$ (first order accurate)

$$u_x(x_j, t) \approx \frac{1}{h}\left(U_{j+1} - U_j\right)$$

We get the following methods:

$$U_j^{n+1} = U_j^n - \frac{ak}{h}\left(U_j^n - U_{j-1}^n\right)$$ (first order in space and time)

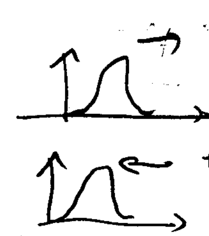
$$U_j^{n+1} = U_j^n - \frac{ak}{h}\left(U_{j+1}^n - U_j^n\right)$$

which method we should use depends on the sign of a.
We know the true solution satisfies:

$$u(x_j, t+k) = u(x_j - ak, t)$$

So when we advance time:

- $a > 0$ U_j takes values on left of U_j
- $a < 0$ ——————— right of U_j



right going pulse

left going pulse

So it makes sense to choose:

if $a>0$ $\quad U_j^{n+1} = U_j^n - \frac{ak}{h}\left(U_j^n - U_{j-1}^n\right)$

if $a<0$ $\quad U_j^{n+1} = U_j^n - \frac{ak}{h}\left(U_{j+1}^n - U_j^n\right)$

(these are incidentally the choices that preserve causality hence "UPWIND")

Stability analysis we already did it! The stability analysis can be obtained from that of Lax - Friedrichs since we can rewrite the method as:

$a>0$ $\quad U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{ak}{2h}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$

$a<0$ $\quad U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) - \underbrace{\frac{ak}{2h}}_{=\frac{\varepsilon}{h^2}}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$

where $\varepsilon = \frac{ah}{2}$.

We need for stability that:

$\left|\frac{ak}{h}\right| \leq 1$ and $-2 < -\frac{2\varepsilon k}{h^2} < 0$

sign depends on a, so last inequality may not be satisfied (which is what happens if you apply upwind in the wrong direction)

To summarize: upwind, Lax Wendroff and Lax Friedrichs all have form:

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{\varepsilon}{h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

where:

$$\varepsilon_{LW} = \frac{a^2 k}{2}$$

$$\varepsilon_{up} = \frac{ah}{2}$$

$$\varepsilon_{LF} = \frac{h^2}{2k}$$