Multidimensional problems

\[ u \delta t = u_{xx} + u_{yy} \quad + \text{I.C. & B.C.} \]

Space discretization: use e.g. 5 point Laplacian:

\[ \Delta h U_{ij} = \frac{1}{h^2} \left[ U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j+1} - 4 U_{ij} \right] \]

Time discretization: use e.g. trapezoidal rule

\[ \rightarrow \text{get Crank-Nicolson} \]

\[ U_{ij}^{n+1} = U_{ij}^n + \frac{h}{2} \left[ \Delta h U_{ij}^n + \Delta h U_{ij}^{n+1} \right] \]

This is an implicit method as can be seen by recasting in matrix form:

\[ \left( I - \frac{h}{2} \Delta h \right) U_{ij}^{n+1} = \left( I + \frac{h}{2} \Delta h \right) U_{ij}^n \]

So we need to solve a system at each step. Since the system is sparse one could use a sparse direct solver. In this case the \( LU \) factors could be computed once and used in all iterations. However the systems may change at each time step (time varying coeff) and sparse direct solvers do not take advantage of previous time step (a good approx to new time step).

Let's look at system matrix more closely:

\[ A = I - \frac{h}{2} \Delta h \]

Eigenvalues are:

\[ \lambda_{pq} (A) = 1 - \frac{h}{h^2} \left[ (\cos(\pi p h) - 1) + (\cos(\pi q h) - 1) \right] \]

\[ p, q = 1, 2, \ldots, m \]
The condition number of $A$ is defined as the ratio of the largest to the smallest eigenvalue of $A$. In this case:

\[
\text{largest eigenvalue of } A = O \left( \frac{1}{k} \right) \text{ (in magnitude)}
\]

\[
\text{smallest eigenvalue of } A = 1 + O(k) \quad \text{(since } \cos(p\pi) - 1 = o \text{ when } p = 1 \text{ and then)}
\]

\[
\cos(p\pi) - 1 \approx \frac{(p\pi)^2}{2}
\]

Thus, $\text{cond}(A) \approx \frac{1}{k} \quad \text{which well conditioned system}$

and we use $U^n_{ij}$ as initial guess $\rightarrow$ iterative methods can with a couple of steps give record order accuracy

(Crank–Nicholson is $O(k^2 + h^2)$ accurate)

There are other ways of avoiding solution of this system that are based on "splitting" or "decoupling" directions.

- **Locally one dimensional (LOD) method:**

Temporary intermediate step:

\[
U^{n+1}_{ij} = U^{n}_{ij} + \frac{k}{2} \left( D^2_{x} U^{n}_{ij} + D^2_{x} U^{n}_{ij} \right)
\]

\[
U^{n+1}_{ij} = U^{n}_{ij} + \frac{k}{2} \left( D^2_{y} U^{*}_{ij} + D^2_{y} U^{*}_{ij} \right)
\]

or in matrix form:

\[
(I - \frac{k}{2} D^2_x) U^* = (I + \frac{k}{2} D^2_x) U^n
\]

\[
(I - \frac{k}{2} D^2_y) U^{n+1} = (I + \frac{k}{2} D^2_y) U^*
\]
The idea in LOD is to successively apply Crank-Nicholson in \(x\)-direction, then \(y\)-direction, \(x\)-dir., \(y\)-dir. etc. 

Here \(D_x^2 U_{ij} \approx U_{xx}(x_i, y_j)\) (and similarly for \(D_y^2 U_{ij}\)).

\[
D_x^2 = \frac{1}{h^2} \begin{bmatrix}
T & 0 & 0 \\
0 & T & 0 \\
0 & 0 & T
\end{bmatrix}
\]

where \(T = \begin{bmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2
\end{bmatrix}\)

Block tridiagonal matrix: rows are decoupled from each other → Cheapest to solve.

\[
D_y^2 = \frac{1}{h^2} \begin{bmatrix}
I & I & 0 \\
I & I & 0 \\
0 & I & I
\end{bmatrix}
\]

where \(I = \text{m x m identity}\)

Here columns are decoupled, and diffusion happens only in \(y\)-direction → system is cheap to solve (same complexity as tridiagonal matrix).

Theoretically (if there are no boundary conditions and as \(h \to 0\)) alternating-directions ends up being like finite operator \(D_x^2 + D_y^2 = \Delta\).
Another idea is the Alternating Direction Implicit method (ADI) of Douglas, Rachford & Peaceman:

\[ U_{ij}^n = U_{ij}^{n-1} + \frac{k}{2} \left( D_y U_{ij}^{n-1} + D_z U_{ij}^n \right) \]

\[ U_{ij}^{n+1} = U_{ij}^n + \frac{k}{2} \left( D_z U_{ij}^n + D_y U_{ij}^{n+1} \right) \]

Here new and old time steps have different directions.

Each of the two steps involves diffusion in \( x \) and \( y \) directions. Each step can be shown to be first order accurate in time, however since the two steps are symmetrical their errors cancel and so this method is \( O(k^2) \).

Of course there are many variations to the discretizations we have presented. Different time and space discretizations can be put together.