Multi dimensional problems

$$u_t = u_{xx} + u_{yy} \qquad + \text{I.C. \& B.C.}$$

Space discretization: use e.g. 5 point Laplacian:

$$\Delta_h U_{ij} = \frac{1}{h^2}\left[U_{i-1j} + U_{i+1j} + U_{ij-1} + U_{ij+1} - 4U_{ij}\right]$$

Time discretization: use e.g. trapezoidal rule

$\rightsquigarrow$ get Crank-Nicholson:

$$U_{ij}^{n+1} = U_{ij}^{n} + \frac{k}{2}\left[\Delta_h U_{ij}^{n} + \Delta_h U_{ij}^{n+1}\right]$$

This is an implicit method as can be seen by rewriting in matrix form:

$$\left(I - \frac{k}{2}\Delta_h\right)U_{ij}^{n+1} = \left(I + \frac{k}{2}\Delta_h\right)U_{ij}^{n}$$

So we need to solve a system at each step. Since the system is sparse one could use a sparse direct solver. In this case the LU factors could be computed once and used in all iterations.

However the systems may change at each time step (time varying coeff) and sparse direct solvers do not take advantage of previous time step (a good approx to new time step)

Let's look at system matrix more closely:

$$A = I - \frac{k}{2}\Delta_h$$

eigenvalues are:

$$\lambda_{p,q}(A) = 1 - \frac{k}{h^2}\left[(\cos(p\pi h) - 1) + (\cos(q\pi h) - 1)\right]$$

$$p, q = 1, 2, \ldots, m$$

condition number of $A = \dfrac{\text{largest eig. of } A}{\text{smallest eig of } A}$

in this case:

largest eigenvalue of $A = \mathcal{O}\left(\frac{k}{h^2}\right)$ (in magnitude)
smallest eigenvalue of $A = 1 + \mathcal{O}(k)$ (since $\cos(p\pi h) - 1 \approx 0$ when $p = 1$ and then: $\cos(\pi h) - 1 \approx -\frac{(\pi h)^2}{2}$)

Thus $\boxed{\operatorname{cond}(A) \approx \frac{k}{h^2}}$ rather well conditioned system

and use U_{ij}^{n} as initial guess

→ iterative methods can with a couple of steps give second order accuracy
(Crank-Nicholson is $\mathcal{O}(h^2 + k^2)$ accurate)

There are other ways of avoiding solution of this system that are based on "splitting" or "decoupling" directions.

- Locally one dimensional (LOD) method:

temporary, intermediary step

$$U_{ij}^{*} = U_{ij}^{n} + \frac{k}{2}\left(D_x^2 U_{ij}^{n} + D_x^2 U_{ij}^{*}\right)$$

$$U_{ij}^{n+1} = U_{ij}^{*} + \frac{k}{2}\left(D_y^2 U_{ij}^{*} + D_y^2 U_{ij}^{n+1}\right)$$

or in matrix form:

$$\left(I - \frac{k}{2} D_x^2\right) U^{*} = \left(I + \frac{k}{2} D_x^2\right) U^{n}$$

$$\left(I - \frac{k}{2} D_y^2\right) U^{n+1} = \left(I + \frac{k}{2} D_y^2\right) U^{*}$$

The idea in LOD is to successively apply Crank-Nicholson in x-direction, then y-direction, x-dir, y-dir etc...

Here $D_x^2 U_{ij} \approx U_{xx}(x_i, y_j)$ (and similarly for $D_y^2 U_{ij}$),

$$D_x^2 = \frac{1}{h^2} \underbrace{\begin{bmatrix} T & & & \\ & T & & \\ & & T & \\ & & & \ddots & \\ & & & & T \end{bmatrix}}_{m \text{ blocks}} \Big\updownarrow {\scriptstyle m \text{ blocks}} \qquad \text{where } T = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

block tridiagonal matrix: rows are decoupled from each other; diffusion happens only in x-direction } CHEAP to solve!

$$D_y^2 = \frac{1}{h^2} \underbrace{\begin{bmatrix} -2I & I & & \\ I & -2I & I & \\ & \ddots & \ddots & I \\ & & I & -2I \end{bmatrix}}_{m \text{ blocks}} \Big\updownarrow {\scriptstyle m \text{ blocks}} \qquad \text{where } I = m \times m \text{ identity}$$

here columns are decoupled, and diffusion happens only on y-direction
→ system is cheap to solve (same complexity as tridiag matrix)

theoretically (if there are no boundary conditions and as $h \to 0$)
alternating directions ends up being like full operator $D_x^2 + D_y^2 = \Delta_h$

Another idea is Altenating Direction Implicit method (ADI) of Douglas, Rachford & Peaceman:

$$U_{ij}^{*} = U_{ij}^{n} + \frac{k}{2}\left(D_y^2 U_{ij}^{n} + D_x^2 U_{ij}^{*} \right)$$

$$U_{ij}^{n+1} = U_{ij}^{*} + \frac{k}{2}\left(D_x^2 U_{ij}^{*} + D_y^2 U_{ij}^{n+1} \right)$$

here new and old time steps have different directions each of the two steps involves diffusion in x and y directions. Each step can be shown to be first order accurate in time. however since the two steps are symmetrical their errors (almost) cancel and so this method is $O(k^2)$.

Of course there are many variations to the discretizations we have presented! Different time and space discretizations can be put together.