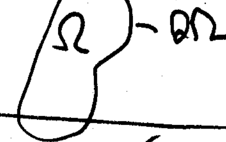
FINITE ELEMENT METHOD RECAP

	1D	2D
STRONG FORM	$\begin{cases} -u'' = f \\ u(0) = u(1) = 0 \end{cases}$	$-\Delta u = f$ $u\vert_{\partial\Omega} = 0$ 
WEAK FORM	Find $u \in V$ s.t. $a(u,v) = (f,v)$ $\forall v \in V$	
V	$V = \{ v \mid \Vert v\Vert_{L^2}^2 + \Vert v'\Vert^2 < \infty$ and $v(0) = v(1) = 0 \}$	$V = \{ v \mid \Vert v\Vert_{L^2}^2 + \Vert v_x\Vert_{L^2}^2 + \Vert v_y\Vert_{L^2}^2 < \infty$ and $v\vert_{\partial\Omega} = 0 \}$
BILINEAR FORM or ENERGY INNER PROD	$a(u,v) = \int_0^1 u'v'\,dx$	$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx$
(f,v)	$(f,v) = \int_0^1 f v\,dx$	$(f,v) = \int_\Omega f v\,dx$

RITZ-GALERKIN

Abstract justification for finite element method:
Instead of solving WF for all $v \in V$, restrict search to a finite dim. subspace

$$V_h \subset V:$$

$$\boxed{\text{Find } u_h \in V_h \text{ s.t. } a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h} \qquad (RG)$$

Given a basis $\{\phi_i\}_{i=1}^n$ of V_h,
let $u_h = \sum_{i=1}^n U_i \phi_i$, $U = \begin{bmatrix} U_1 \\ \vdots \\ U_n \end{bmatrix}$, $F = \begin{bmatrix} (f,\phi_1) \\ (f,\phi_2) \\ \vdots \\ (f,\phi_n) \end{bmatrix}$

and $A = [a(\phi_i, \phi_j)]_{i,j=1}^n \in \mathbb{R}^{n\times n}$ = <u>stiffness matrix</u>

We have that:

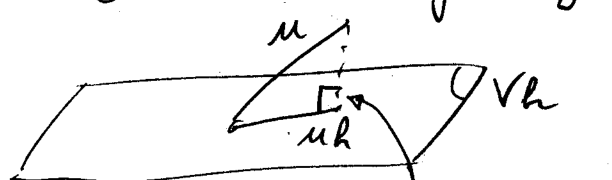
$$(RG) \iff A\,U = F \quad \text{(linear system)}$$

Finite element method gives a systematic way of constructing bases for V_h with guaranteed convergence behavior and that are relatively easy to solve (A is sparse).

Fundamental error estimate:

$$\left.\begin{aligned} a(u, v_h) &= (f, v_h) \\ a(u_h, v_h) &= (f, v_h) \end{aligned}\right\} \Rightarrow \quad a(u-u_h, v_h) = 0 \quad \forall v_h \in V_h$$

"Galerkin orthogonality"



$\perp$ in the sense of inner prod $a(\cdot,\cdot)$

$$\begin{aligned} \|u-u_h\|_E^2 &= a(u-u_h, u-u_h) \\ &= a(u-u_h, u-v_h) + \underbrace{a(u-u_h, v_h-u_h)}_{=0 \text{ by Galerkin } \perp} \\ &\leq \|u-u_h\|_E \|u-v_h\|_E \end{aligned}$$

$$\Rightarrow \quad \|u-u_h\|_E \leq \|u-v_h\|_E \quad \forall v_h \in V_h$$

meaning: $\boxed{\text{Ritz Galerkin gives best possible solution in } V_h \text{ (in the sense of energy norm)}}$

Finite Element

$(K , \mathcal{P} , \mathcal{N})$

K	$\mathcal{P}$	$\mathcal{N}$
finite element spatial support	approx. space, shape functions, <u>usually</u> $\mathcal{P} = P_k$ = poly of degree $\leq k$. $d = \dim P_k$	nodal values, degrees of freedom = linear functionals $\mathcal{P} \to \mathbb{R}$ that can completely determine $\mathcal{P}$ $\mathcal{N} = \{N_1, \dots, N_d\}$ s.t. $p \in \mathcal{P}_k$ with $N_1(p) = \dots = N_d(p) = 0$ $\Rightarrow p = 0$.

For convenience we choose a basis $\{\phi_i\}_{i=1}^{d}$ of P_k s.t.

$$N_i(\phi_j) = \delta_{ij} \quad , \; i,j = 1..d.$$

Any $p \in P_k$, can be written as:

$$p = \sum_{i=1}^{d} N_i(p)\, \phi_i = I_K\, p$$

<u>Examples</u>:

$([0,1], P_1, \{N_1, N_2\})$ $\quad N_1(v) = v(0)$, $N_2(v) = v(1)$ — any $p \in P_1$ is uniquely determined by its values at 2 points (0 and 1)

$([0,1], P_k, \{N_0, N_1, \dots, N_k\})$ $\quad N_j(v) = v(j/k)$, $j = 0, \dots, k$

any $p \in P_k$ is uniquely determined by its values at $k+1$ points.

$\{\phi_i\}_{i=0}^{k}$ are Lagrange interp. poly.

$([0,1], P_3, \{N_0, N_1, N_2, N_3\})$

$N_0(v) = v(0)$
$N_1(v) = v'(0)$
$N_2(v) = v(1)$
$N_3(v) = v'(1)$

with these elements one can construct $V_h \subset C^1$

Construction of V_h

Simply require that if $f \in V_h$, then $f|_K \in \mathbb{P}_k$

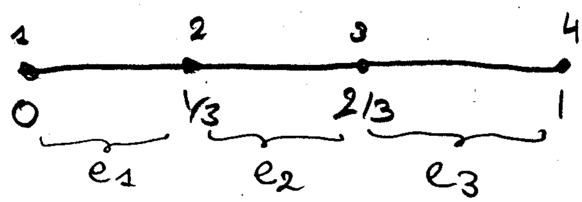
i.e. V_h is the space of functions in V that are piecewise polynomials over the elements.

$\Rightarrow$ Having easy to handle polynomials in each element makes it easier to construct stiffness matrix A and right hand side F element by element.

<u>local to global map</u> maps a local d.o.f. of an element to the global d.o.f. index.

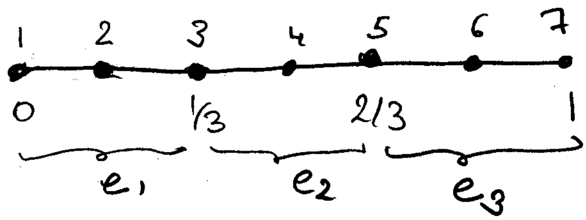
<u>Examples</u>:

- <u>1D P1 elements</u>



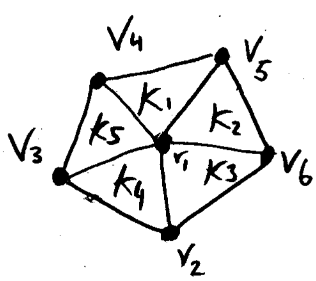
$$i = \begin{array}{c} \text{\# loc. d.o.f.} \\ \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \text{\# elements} \end{array}$$

- <u>1D, P2 elements</u>



$$i = \begin{array}{c} \text{\# loc. d.o.f.} \\ \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} \text{\# elements} \end{array}$$

- <u>2D, P1 triangular elements</u>



$$i = \begin{array}{c} \text{\# loc. d.o.f.} \\ \begin{bmatrix} 1 & 5 & 4 \\ 6 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix} \text{\# elt.} \end{array}$$

vertices of K_5

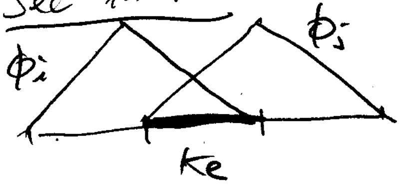
Implementation of finite elements:

$$a(\phi_i, \phi_j) = \sum_e a_e(\phi_i, \phi_j)$$

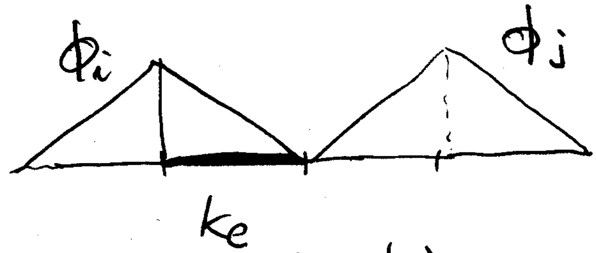
where $a_e(u,v) = \int_{K_e} \nabla\phi_i \cdot \nabla\phi_j \, dx$ (in 2D)

it is clear that $a_e(\phi_i, \phi_j) \neq 0$ only if i and j are d.o.f. belonging to the same element.

See in 1D:



$a_e(\phi_i,\phi_j) \neq 0$



$a_e(\phi_i, \phi_j) = 0$

(→ this is what makes stiffness matrix A sparse.)

Also $\phi_{i(e,j)}|_{K^e} = \phi_j^{K^e}$ = local basis function
= polynomial, so integrals are relatively easy to compute

So we assemble stiffness matrix A and R.H.S F as follows:

> A = all zero # d.o.f × # d.o.f. matrix (preferably sparse)
> F = all zero # d.o.f × 1 vector
> for $e = 1 ..$ # elements
> | $A(i(e,:), i(e,:)) = A(i(e,:), i(e,:)) + A_{loc}^{K^e}$
> | $F(i(e,:)) = F(i(e,:)) + F_{loc}^{K_e}$

where
$$\left(A_{loc}^{K_e}\right)_{i,j} = a_{K_e}(\phi_i^e, \phi_j^e), \quad i,j = 1 .. \# \text{loc. d.o.f.}$$
$$\left(F_{loc}^{K^e}\right)_i = \int_{K_e} f|_{K_e} \, \phi_i^e \, d\underline{x}, \quad i = 1 .. \# \text{loc d.o.f.}$$

Error Estimates

Here are the main ingredients:

① RITZ GALERKIN BASIC ERROR ESTIMATE:

$$\|u-u_h\|_E \leq \|u-v_h\|_E \quad \forall v_h \in V_h$$
$$\leq \|u-\mathcal{I}_{\mathcal{T}_h} u\|_E \quad (\text{interpolation error})$$

② CEA's Lemma

If bilinear form $a(u,v)$ satisfies: $\forall u, v \in V$:

i) $a(u,u) \geq \alpha \|u\|^2_{H^1(\Omega)}$ (coercivity)

ii) $a(u,v) \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$ (continuity)

then: $\|\bullet\|_E$ is equivalent to $\|\bullet\|_{H^1}$ and:

$$① \Rightarrow \|u-u_h\|_{H^1} \leq C\|u-v_h\|_{H^1} \quad \forall v_h \in V_h$$
$$\leq \|u-\mathcal{I}_{\mathcal{T}_h} u\|_{H^1}$$

③ Interpolation error if we use P_k as shape functions:

$$\|u-\mathcal{I}_{\mathcal{T}} u\|_{H^1(\Omega)} \leq C h^k |u|_{H^{k+1}(\Omega)}$$

$$①+②+③ \Rightarrow \boxed{\|u-u_h\|_{H^1(\Omega)} \leq C h^k |u|_{H^{k+1}(\Omega)}}$$

④ Duality argument (AUBIN-NITSCHE Lemma) (p 99-100)

$$\|u-u_h\|_{L^2(\Omega)} \leq C h \|u-u_h\|_{H^1(\Omega)}$$

$$①+②+③+④ \Rightarrow \boxed{\|u-u_h\|_{L^2(\Omega)} \leq C h^{k+1} |u|_{H^{k+1}(\Omega)}}$$

(Consistent with Taylor: for $u_h \in P_k$ then $u-u_h = O(h^{k+1})$)