Error Estimates

Now we put everything together to get a priori error estimates.

① FEM error and interp error (p95)

$$\|u-u_h\|_E \leq \inf_{v_h \in V_h} \|u-v_h\|_E \leq \|u - \mathcal{I}_{Z_h} u\|_E$$

In fact under some assumptions on $a(u,v)$ namely:

i) $a(u,u) \geq \alpha \|u\|^2_{H^1(\Omega)}$ (Coercivity)

ii) $a(u,v) \leq C\|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$ (continuity).

the norms $\|\cdot\|_E$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent meaning $\exists\, c_1, c_2 > 0$ s.t.

$$c_1\|u\|_{H^1(\Omega)} \leq \|u\|_E \leq c_2 \|u\|_{H^1(\Omega)}$$

thus we can show $\exists C > 0$ s.t.

$$\boxed{\|u-u_h\|_{H^1(\Omega)} \leq C \inf_{v_h \in V_h} \|u-v_h\|_{H^1(\Omega)} \leq C\|u - \mathcal{I}_{Z_h} u\|_{H^1(\Omega)}}$$

This is called in FEM speak "CEA's LEMMA".

② Interp error (p96)

If we use P_k for our shape functions ($k \geq 1$) on each triangle:

$$\|u - \mathcal{I}_Z u\|_{H^1(\Omega)} \leq C h^k |u|_{H^{k+1}(\Omega)}$$

Combining both results we get:

$$\boxed{\|u-u_h\|_{H^1(\Omega)} \leq C h^k |u|_{H^{k+1}(\Omega)}}$$

Thus the highest the polynomial degree the higher convergence rate, provided the solution is smooth enough ($|u|_{H^{k+1}(\Omega)}$ could also be large, in which case the estimate could be useless in practice!)

Error estimates in the $L^2(\Omega)$ norm

We shall use a duality argument (AUBIN - NITSCHE's lemma) to show that: (see also p 79)

$$\|u-u_h\|_{L^2(\Omega)} \leq C h \|u-u_h\|_{H^1(\Omega)}$$

Therefor the error estimate in the $L^2(\Omega)$ norm is:

$$\boxed{\|u-u_h\|_{L^2(\Omega)} \leq C h^{k+1} |u|_{H^{k+1}(\Omega)}}$$

To do duality argument, let $e = u - u_h$ and: w s.t.

$$\begin{cases} -\Delta w = e & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

Variational (weak) formulation of this problem is:

Find $w \in V$ s.t. $a(w, v) = (e, v) \quad \forall v \in V$

Therefore:

$$\|u-u_h\|^2_{L^2(\Omega)} = (\underbrace{u-u_h}_{=e}, \underbrace{u-u_h}_{=\text{test fun.}})$$
$$= a(w, u-u_h)$$
$$= a(w-\mathcal{I}_h w, u-u_h) \quad \left(\begin{array}{l}\text{Galerkin } \perp \text{ of}\\ \text{error } u-u_h.\end{array}\right)$$
$$\leq C\,\|w-\mathcal{I}_h w\|_{H^1(\Omega)}\,\|u-u_h\|_{H^1(\Omega)} \quad \left(\begin{array}{l}\text{continuity of}\\ \text{bilinear form}\end{array}\right)$$
$$\leq Ch\,\|u-u_h\|_{H^1(\Omega)}\,|w|_{H^2(\Omega)}$$
$$\leq Ch\,\|u-u_h\|_{H^1(\Omega)}\,\|u-u_h\|_{L^2(\Omega)}$$

Using an elliptic regularity estimate:

$$|w|_{H^2(\Omega)} \leq C\,\|e\|_{L^2(\Omega)}$$

(Roughly saying $|w|_{H^2(\Omega)} \leq C\|\Delta w\|_{L^2(\Omega)} = C\|e\|_{L^2(\Omega)}$)

which shows the desired result: $\|u-u_h\|_{L^2(\Omega)} \leq Ch\,\|u-u_h\|_{H^1(\Omega)}$