How do we identify a basis for $\mathcal{P}'$?

Lemma

$\{N_1, N_2, \dots, N_d\} \subset \mathcal{P}'$, with $\dim \mathcal{P} = d$. is a basis for $\mathcal{P}'$ iff:

$$\forall v \in \mathcal{P}, \quad N_i v = 0 \text{ for } i = 1, 2, \dots d \Rightarrow v = 0$$

For example for P1 1D elements:

$$v \in P_1 \Rightarrow v = ax + b$$

$$N_1(v) = N_2(v) = 0 \quad \Rightarrow a = b = 0$$
$$\Rightarrow v = 0.$$

Definition: We say that $\mathcal{N}$ determines $\mathcal{P}$ if $\psi \in \mathcal{P}$ with

$$N(\psi) = 0 \ \forall N \in \mathcal{N} \Rightarrow \psi = 0.$$

TRIANGULAR FINITE ELEMENTS : K = triangle

$\mathcal{P} = P_k$ where P_k = polynomials of x, y w/ degree $\leq k$.

$P_1 = \text{span}\{1, x, y\}$ $\quad$ $\dim P_1 = 3$

$P_2 = \text{span}\{1, x, y, xy, x^2, y^2\}$ $\quad$ $\dim P_2 = 6$

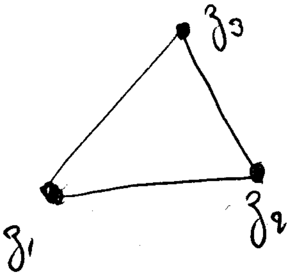
$P_3 = \text{span}\{1, x, y, xy, x^2, y^2, xy^2, x^2y, x^3, y^3\}$ $\quad$ $\dim P_3 = 10$

etc...

<u>Case $k=1$</u>: Let $\mathcal{N}_1 = \{N_1, N_2, N_3\}$

(note we take $\dim \mathcal{N}_1 = \dim \mathcal{P}_1$)

where $N_i(v) = v(z_i)$ and z_i = vertices of triangle K

(*) 

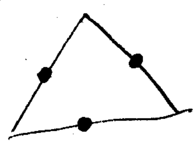
Lagrange triangle, Courant element.

Can we show $\mathcal{N}_1$ determines P_1?

Let $v \in P_1$. $N_1(v) = N_2(v) = 0 \Rightarrow v|_{[z_1, z_2]} = 0$

and same for all segments we see that $v = 0$.

Note: This choice is not unique, but is the most popular. Depending on application it may make more sense to use Crouzeix-Raviart elements:



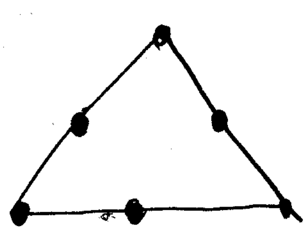
$\rightsquigarrow$ global space will be continuous at the midpoints but allows for discont at vertices of triangles.

Also other linear functionals like integrals over edges, directional derivatives can be consid

We will only work with element (*) (P1 triangular element)

But you should be aware that higher order polynomials can be used inside each triangle:

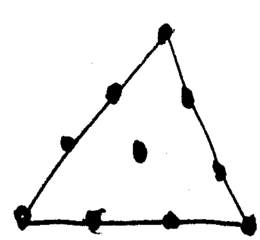
P_2



$v|_{edge}$ is a quadratic and continuity is assured on each edge.

6 dofs = $\dim P_2$

P_3



$v|_{edge}$ = cubic and continuity is achieved w/ neighboring element.

10 dofs = $\dim P_3$

We now define the <u>local interpolant</u>:

$$I_K v = \sum_{i=1}^{k} N_i(v)\,\phi_i$$

where $N_i \in \mathcal{N}$, $\phi_i \in \mathcal{P}$ for dual basis

<u>Note</u>: I_K is <u>linear</u> and is a projector onto $\mathcal{P}$.

$$N_i(I_K(v)) = N_i(v)$$

$$I_K^2 v = I_K v$$

We now piece the elements together and see what finite dim space we are approximating with.

<u>Def</u>: A <u>subdivision</u> of a domain Ω is a finite collection of element domains $\{K_i\}$ s.t.

i) $\mathring{K}_i \cap \mathring{K}_j = \emptyset$ if $i \neq j$

(i.e. two elements may share only their boundary)

ii) $\bigcup K_i = \overline{\Omega}$

(i.e. all elements cover the domain)

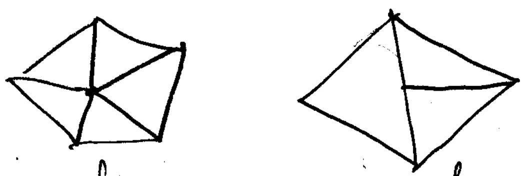
We can then extend the notion of interpolant to the subdivision.

The <u>global interpolant</u> on a subdivision $\mathcal{T}$ is:

$$I_{\mathcal{T}} f|_{K_i} = I_{K_i} f$$

If we are to say something about the continuity of the global interpolant we need to be more precise about subdivision.

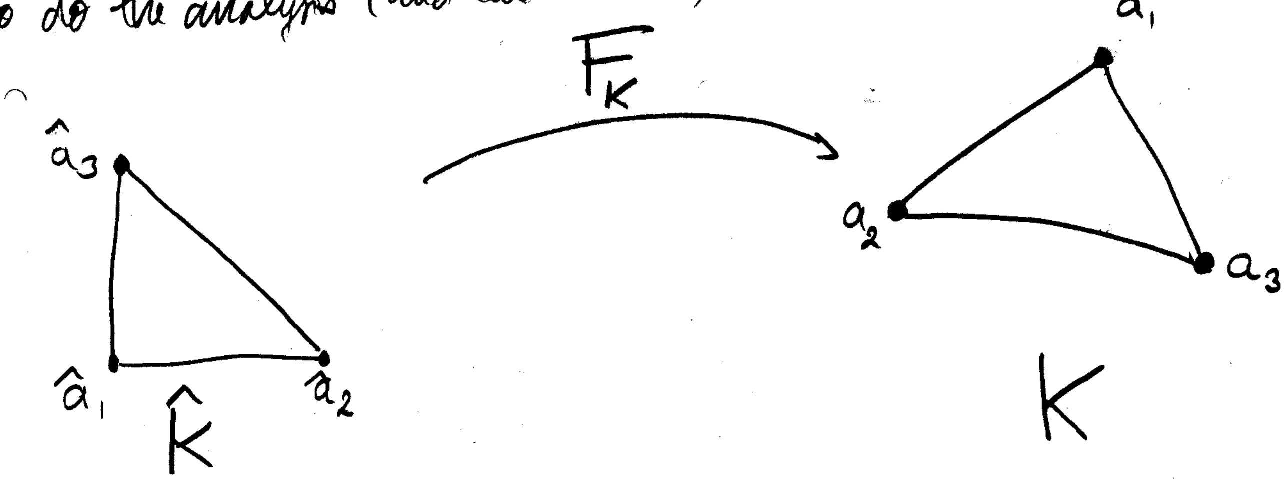
A <u>triangulation</u> of a domain Ω is a subdivision composed of triangles where no vertex of any triangle lies in the interior of an edge:



ok not ok.

Then it is possible to show $\mathcal{I}_{\mathcal{T}} f \in C^0$ when Lagrange elements are used.

<u>Affine equivalence of elements</u>: later we will need some bounds on $\|\mathcal{I}_K\|$ that are independent of the element K. Therefore it is convenient to look at a canonical (parent) element to do the analysis (and calculations).



$F_K(\hat{x}) = A\hat{x} + b$ = affine mapping

$(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ and $(K, \mathcal{P}, \mathcal{N})$ are <u>affine equiv</u>

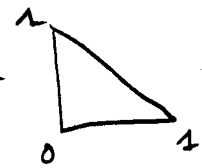
if there is an invertible affine mapping F_K s.t.

i) $F_K(\hat{K}) = K$

ii) $\forall\, p \in \mathcal{P}: \quad p = \hat{p} \circ F_K^{-1}$, where $\hat{p} \in \hat{\mathcal{P}}$.

iii) $\forall\, N \in \mathcal{N}: \quad N(v) = \hat{N}(v \circ F_K)$, where $\hat{N} \in \hat{\mathcal{N}}$.

<u>Example</u>: $(K, \mathcal{P}_1, \mathcal{N})$ where $\mathcal{N}$ = eval at vertices of triangle:

are all affine equivalent to: $\hat{K} =$  since:

- clearly there is an affine mapping s.t. $F_K(\hat{K}) = K$ and:
- if $\hat{p} \in P_1$ then $p(x) = \hat{p}(F_K^{-1}(x)) \in P_1$.
- $$N_1(v) = v(a_1) = v(F_K(\hat{a}_1)) = \hat{N}_1(v \circ F_K)$$

etc...

Interpolation results:

Why do we need interpolation results?

Given a problem such as

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

← models steady state heat eq. or conduction of electricity.

we will write it in variational (weak) form as:

Find $u \in V$ s.t. $a(u,v) = (f,v) \quad \forall v \in V$

where $V = \{ u \mid \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 < \infty$ and $u|_{\partial\Omega} = 0 \}$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

$$(f,v) = \int_\Omega f v \, dx$$

Then use Galerkin method:

$$V_h = \{ u \in V \mid u|_K \in P_K \}$$

Find $u_h \in V_h$ s.t. $a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$.

Then we must have by Galerkin orthogonality:

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$

Thus:

$$\|u - u_h\|_E^2 = a(u-u_h, u-u_h) \leq a(u-u_h, u-v_h) + \underbrace{\cancel{a(u-u_h, v_h-u_h)}}_{=0}$$

$$\leq \|u-u_h\|_E \, \|u-v_h\|_E \quad \forall v_h \in V_h$$

Thus we get the basic estimate.

$$\|u-u_h\|_E \leq \inf_{v_h \in V_h} \|u-v_h\|_E \leq \underbrace{\|u-\mathcal{I}_{\tau} u\|_E}$$

we want to estimate this quantity or determine how well does the interpolant approx u.

To study this question we borrow results from the theory of interpolation in Sobolev spaces, and we first find estimates of the errors over each element:

$\boxed{\mathcal{I}_K p = p \quad \forall p \in P^k}$

$$\forall v \in H^{k+1}(K) \quad \|v-\mathcal{I}_K v\|_{H^m(K)} \leq C\, h_K^{k+1-m} \left[\sum_{|\alpha|=k+1} \|D^\alpha v\|^2_{L^2(K)}\right]^{1/2}$$

(LOC) for $0 \leq m \leq k+1$

k ← degree of poly.

here we used the notation: $h_K = \mathrm{diam}(K)$

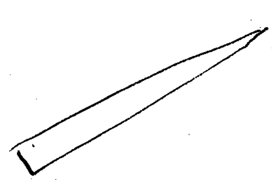
$$H^m(K) = \left\{ v \;\middle|\; \|v\|^2_{H^m(K)} = \sum_{|\alpha| \leq m} \|D^\alpha f\|^2_{L^2(K)} < \infty \right\}$$

$$D^\alpha f = \frac{\partial^{\alpha_1} \partial^{\alpha_2} \cdots \partial^{\alpha_n} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \dots + \alpha_n$$

(multi index notation of derivatives)

This result holds for affine families of regular finite elements.

Regularity means there are no unpleasant surprises as $h = \max_{K \in \tau_h} h_K \to 0$ such as elements that become degenerate



When we use triangles, regularity can be achieved by imposing a minimal angle condition due to ZLÁMAL:

$\boxed{\forall K \quad \theta_K \geq \theta_0 > 0}$, where θ_K = min angle of K.

so triangles that do not become degenerate with refinement of triangulation are important (that's why using a good grid generator is good).

<u>How do we go from a local interpolation result to a global one?</u>

Simply if (LOC) holds then $\forall v \in H^{k+1}(K)$

$$\|v - \mathcal{I}_{\mathcal{T}} v\|_{H^m(\Omega)} = \left[\sum_{K \in \mathcal{T}_h} \|v - \mathcal{I}_K v\|^2_{H^m(K)} \right]^{\frac{1}{2}}$$

I am assuming polygonal domains here.

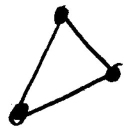
$$\underset{(LOC)}{\leq} \left[\sum_{K \in \mathcal{T}_h} |v|^2_{H^m(K)} \right]^{\frac{1}{2}} C h^{k+1-m}$$

$$= C h^{k+1-m} |v|_{H^m(\Omega)}$$

where we have used the notation:

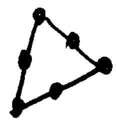
$$|v|^2_{H^m(\Omega)} = \sum_{|\alpha| = m} \|D^\alpha v\|^2_{L^2(\Omega)}$$

Examples:



$k=1$ $\quad \| v - I_K \|_{H^m(K)} \leq C h^{2-m} |v|_{H^2(K)}$

$0 \leq m \leq 2$



$k=2$ $\quad \| v - I_K \|_{H^m(K)} \leq C h^{3-m} |v|_{H^3(K)}$

$0 \leq m \leq 3$

$k=3$ $\quad \| v - I_K \|_{H^m(K)} \leq C h^{4-m} |v|_{H^4(K)}$

$0 \leq m \leq 4$

Therefore increasing degree of polynomials inside elements gives an additional order k convergence in general