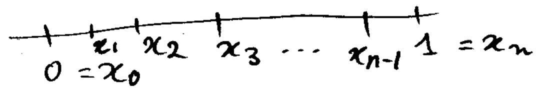
The Finite element method

Piecewise polynomial spaces:

recall problem

$$\begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

partition $[0,1]$:



Let S be the linear space of functions v s.t.

i) $v \in C^0[0,1]$

ii) $v|_{[x_{i-1}, x_i]}$ is a linear poly, $i = 1..n$

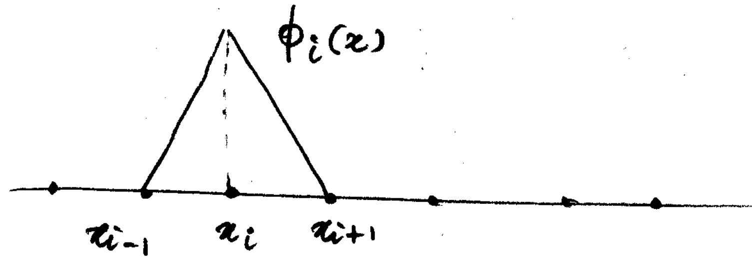
iii) $v(0) = 0$

Recall $V = \{ v \in L^2[0,1] \mid \int_0^1 (v')^2 dx < \infty \text{ and } v(0) = 0 \}$,

It's not hard to see $S \subset V$ (since v' is def. a.e. as a pcwis const. function)

We define the functions ϕ_i, $i = 1..n$ with the requirement:

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} = \text{"Kronecker" delta.}$$



<u>Lemma</u>: $\{\phi_i\}_{i=1}^n$ is a basis for S.

$\{\phi_i\}_{i=1}^n$ is called a <u>nodal</u> basis for S

$\{v(x_i)\}_{i=1}^n$ = <u>nodal values</u> of a function v

$\{x_i\}_{i=1}^n$ = <u>nodes</u>

To show $\{\phi_i\}_{i=1}^{n}$ is a basis for S we show:

- $\{\phi_i\}_{i=1}^{n}$ are linearly indep:

$$\sum_{i=1}^{n} c_i \phi_i = 0 \Rightarrow \sum_{i=1}^{n} c_i \phi_i(x_j) = 0 \Rightarrow c_j = 0 \quad \text{for } j = 1 \dots n.$$

- span $\{\phi_i\}_{i=1}^{n} = S$:

To show this it is convenient to introduce the interpolant $v_I \in S$ of v:

$$v_I = \sum_{i=1}^{n} v(x_i)\, \phi_i$$

Then if we could show that $\forall v \in S \quad v = v_I$, then we would have that the ϕ_i span S.

This is true since: if $v \in S$, $v - v_I$ is linear on each $[x_{i-1}, x_i]$ and by construction zero at both end points, so $v - v_I = 0$.

A simple interpolation result:

Let $h = \max_{1 \le i \le n} (x_i - x_{i-1})$ then $\forall u \in V$:

$$\boxed{\|u - u_I\|_E \leq C h \|u''\|}$$

Here C is a constant independent of h and u.

proof It is sufficient to prove the estimate piecewise that is:

$$\int_{x_{j-1}}^{x_j} [(u-u_I)'(x)]^2 dx \leq c\,(x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} [u''(x)]^2 dx$$

Since summing over j gives:

$$\|u - u_I\|_E^2 = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} [(u-u_I)']^2 dx \leqslant c \sum_{j=1}^{n} (x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} [u'']^2 dx \leq c h^2 \|u''\|.$$

where $c = C^2$, C being the constant in statement.

Let $e = u - u_I$ then the inequality we want to prove is equiv to:

$$\int_{x_{j-1}}^{x_j} [e'(x)]^2 dx \le c\,(x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} [e''(x)]^2 dx$$

(simply because $e''(x) = u'' - u_I'' = u''$ as $u_I =$ linear)

Then we change variables so that we map element $[x_{j-1}, x_j]$ to $[0,1]$ (this is a recurrent theme in finite elements!).

$$x(\tilde{x}) = x_{j-1} + \underbrace{\tilde{x}}_{\text{new variable } \in [0,1]}(x_j - x_{j-1})$$

$$\tilde{e}(\tilde{x}) = e(x(\tilde{x})) = e\,(x_{j-1} + \tilde{x}(x_j - x_{j-1}))$$

$$\cancel{(x_j - x_{j-1})} \int_0^1 [e'(x(\tilde{x}))]^2 d\tilde{x} \le c\,(x_j - x_{j-1})^2\, \cancel{(x_j - x_{j-1})} \int_0^1 [e''(x(\tilde{x}))]^2 d\tilde{x}$$

But by the chain rule:

$$\tilde{e}'(\tilde{x}) = (x_j - x_{j-1})\, e'(x(\tilde{x}))$$

$$\tilde{e}''(\tilde{x}) = (x_j - x_{j-1})^2\, e''(x(\tilde{x}))$$

Thus:

$$\cancel{\frac{1}{(x_j - x_{j-1})^2}} \int_0^1 [\tilde{e}'(\tilde{x})]^2 d\tilde{x} \le c\, \cancel{\frac{1}{(x_j - x_{j-1})^4} (x_j - x_{j-1})^2} \int_0 [\tilde{e}''(\tilde{x})]^2 d\tilde{x}.$$

Now we need to check estimate:

$$\int_0^1 [\tilde{e}'(\tilde{x})]^2 d\tilde{x} \le c \int_0^1 [\tilde{e}''(\tilde{x})]^2 d\tilde{x}$$

which is an analysis exercise, we shall do this with a different notation for clarity:

$$\int_0^1 (w')^2(x)\,dx \le c \int_0^1 (w'')^2(x)\,dx$$

First note that $w(0) = w(1) = 0$ since the interpolant coincide with function on all nodes.

$\Rightarrow$ Use Rolle's (or MVT): $\exists\, \xi \in (0,1)$ s.t. $w'(\xi) = 0$:

$$w'(y) - \underbrace{w'(\xi)}_{=0} = \int_{\xi}^{y} w''(x)\,dx$$

By Schwarz:

$$|w'(y)| = \left|\int_{\xi}^{y} w''(x)\,dx\right| = \left|\int_{\xi}^{y} 1 \cdot w''(x)\,dx\right|$$

$$\leq \underbrace{\left|\int_{\xi}^{y} 1\,dx\right|^{\frac{1}{2}}}_{=|y-\xi|^{\frac{1}{2}}} \underbrace{\left|\int_{\xi}^{y} [w''(x)]^2\,dx\right|^{\frac{1}{2}}}_{\leq\left[\int_0^1 [w''(x)]^2 dx\right]^{\frac{1}{2}}}.$$

Squaring and integrating we get result since

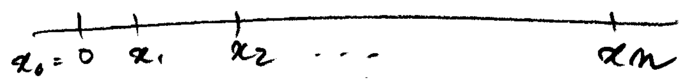
$$\max_{0 \leq \xi \leq 1} \int_0^1 |y - \xi|\,dy = \frac{1}{2}$$

Computer implementation of the finite element method

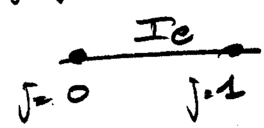
Key idea: Compute $K = \left[a(\phi_i,\phi_j)\right]_{i,j=1}^{n}$ by summing the local contributions of each element.

We need a local-to-global index $i(e,j)$ that relates the local degree of freedom j on element e to its global index e.

Recall our 1D example:



The elements are $I_e = [x_{e-1}, x_e]$, $e = 1..n$.
Each element has 2 degrees of freedom (the nodes of interval).



and in order to get the global index associated with node j on element e we need:

$$\boxed{i(e,j) = e + j + 1, \quad e = 1..n, \quad j = 0, 1.}$$

(of course this relation depends on particular triangulation you make, and is usually more complicated than this).

We may rewrite the interpolant of a continuous function f for the space of all piecewise linear fun on grid with:

$$f_I = \sum_{i=0}^{n} f(x_i)\,\phi_i = \sum_{e} \sum_{j=0}^{1} f(x_{i(e,j)})\,\phi_j^e$$

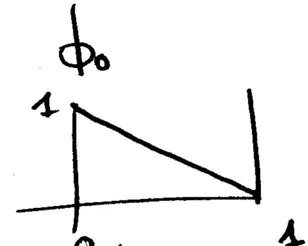
note: this expression gets values at x_i wrong (added twice) but this does not matter when computing integrals!

where $\{\phi_j^e\}_{j=0}^{1}$ are the basis functions for linear functions on the single interval $I_e = [x_{e-1}, x_e]$.

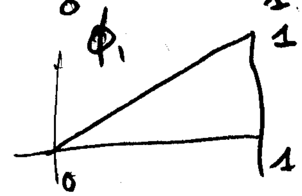
$$\phi_j^e(x) = \phi_j\left(\frac{x - x_{e-1}}{x_e - x_{e-1}}\right)$$

where

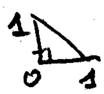
$$\phi_0(x) = \begin{cases} 1-x & x \in [0,1] \\ 0 & \end{cases}$$



$$\phi_1(x) = \begin{cases} x & x \in [0,1] \\ 0 & \end{cases}$$



Note (this will happen in 2D too) we have related the "local" basis fun ϕ_j^e to those at a reference (parent) element $[0,1]$ via an affine mapping. $[0,1] \longrightarrow [x_{e-1}, x_e]$

(in 2D  $\rightarrow$  arbitrary triangle in triangulation)

Therefore in order to construct bilinear form $a(u,v)$:

$$a(u,v) = \sum_e a_e(u,v)$$

where the "local bilinear form" is:

$$a_e(u,v) = \int_{I_e} u'v'\,dx$$

$$= \frac{1}{|I_e|}\int_0^1 \Big(\sum_{j=0}^{1} u_{i(e,j)}\,\phi_j\Big)'\Big(\sum_{j=0}^{1} v_{i(e,j)}\phi_j\Big)'\,dx$$

$$= \frac{1}{|I_e|}\,[v_i(e,0)\;\; v_i(e,1)]\,K_{loc}\begin{bmatrix} v_i(e,0) \\ v_i(e,1)\end{bmatrix}$$

where $(K_{loc})_{ij} = \int_0^1 \phi'_{i-1}\, \phi'_{j-1}\, dx$ for $i,j = 1,2$.
"local stiffness matrix"

The construction of a finite element space

Recall (WF):

Find $u \in V$ $a(u,v) = F(v)$ $\forall v \in V$.

We need to construct a finite dimensional $S \subset V$.
If we look at how we defined such S in D we needed to specify the following:

1. What is the function restricted to an element ($\to$ linear)
2. How is a function determined in an element ($\to$ by its nodal values)
3. How do the restrictions of the function to 2 neighboring elements match at the boundary ($\to$ same nodal value, continuity).

Definition (Finite Element)

A triplet $(K, \mathcal{P}, \mathcal{N})$ is called a <u>finite element</u> if:

i) $K \subset \mathbb{R}^n$ is a bounded closed set with non empty interior and pws smooth bdry. ("element domain")

ii) $\mathcal{P}$ = finite dimensional space of functions on K ("shape functions")

iii) $\mathcal{N} = \{N_1, N_2, \dots, N_k\}$ = basis of dual of $\mathcal{P}$. (d.o.f.)

Some Hilbert space theory

$\mathcal{P}'$ = dual of $\mathcal{P}$ = Vector space of all linear functionals of $\mathcal{P}$

$\dim \mathcal{P}' = \dim \mathcal{P}$ (when $\mathcal{P}$ is finite dim)

(a linear functional is a linear mapping $\mathcal{P} \to \mathbb{R}$)

<u>Def (nodal basis)</u> Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. The basis $\{\phi_1, \phi_2, \ldots, \phi_k\}$ of $\mathcal{P}$ s.t.

$$N_i(\phi_j) = \delta_{ij}$$

is called nodal basis of $\mathcal{P}$.

<u>Example (1D)</u>: $K = [0,1]$, $\mathcal{P} = P_1$ = linear poly

$\mathcal{N} = \{N_1, N_2\}$ with $N_1(v) = v(0)$
$N_2(v) = v(1)$
$\forall v \in \mathcal{P}$.

$(K, \mathcal{P}, \mathcal{N})$ = finite element

and nodal basis is: $\phi_1(x) = 1 - x$
$\phi_2(x) = x$

however other element are possible: for example P_2 elements with d.o.f. being values at end points of interval and at center.