The Finite element method

Piecewise polynomial spaces:

Partition $[0,1]$:

$0 = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = 1$

Let $S$ be the linear space of functions $v$ s.t.

i) $v \in C^0[0,1]$

ii) $v|_{[x_{i-1}, x_i]}$ is a linear poly, $i = 1, ..., n$

iii) $v(0) = 0$

Recall $V = \{ v \in L^2[0,1] \mid \int_0^1 v^2 dx < \infty \text{ and } v(0) = 0 \}$.

It's not hard to see $S \subset V$ (since $v$ is def. a.e. as a piecewise function).

We define the functions $\phi_i$, $i = 1, ..., n$ with the requirement:

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

**Lemma**: $\{ \phi_i \}_{i=1}^n$ is a basis for $S$.

$\{ \phi_i \}_{i=1}^n$ is called a **nodal basis** for $S$.

$\{ v(x_i) \}_{i=1}^n$ = modal values of a function $v$.

$\{ x_i \}_{i=1}^n$ = node.
To show \( \{ \phi_i \}_{i=1}^n \) is a basis for \( S \) we show:

- \( \{ \phi_i \}_{i=1}^n \) are linearly independent:
  \[
  \sum_{i=1}^n c_i \phi_i = 0 \Rightarrow \sum_{i=1}^n c_i \phi_i(x_j) = 0 \Rightarrow c_j = 0 \text{ for } j=1, \ldots, n.
  \]

- \( \text{span} \{ \phi_i \}_{i=1}^n = S \):
  
  To show this it is convenient to introduce the interpolant \( \mathbf{v}_I \in S \) of \( \mathbf{v} \):
  
  \[
  \mathbf{v}_I = \sum_{i=1}^n \mathbf{v}(x_i) \phi_i.
  \]

  Then if we could show that \( \forall \mathbf{v} \in S \), \( \mathbf{v} = \mathbf{v}_I \), then we would have that the \( \phi_i \) span \( S \).

  This is true since if \( \mathbf{v} \in S \), \( \mathbf{v} - \mathbf{v}_I \) is linear on each \([x_i, x_{i+1}]\) and by construction goes to both end points, so \( \mathbf{v} - \mathbf{v}_I = 0 \).

### Asymptotic interpolation result:

Let \( R = \max (x_i - x_{i-1}) \) then \( \forall \mathbf{v} \in V \):

\[
\|\mathbf{u} - \mathbf{v}_I\|_E \leq CR \| \mathbf{u}'' \|
\]

where \( C \) is a constant independent of \( R \) and \( \mathbf{v} \).

**Proof**

It is sufficient to prove the estimate provided that

\[
\int_{x_{j-1}}^{x_j} [u - u_I]'(x)^2 \, dx \leq C (x_j - x_{j-1}) \int_{x_{j-1}}^{x_j} [u''(x)]^2 \, dx
\]

Since summing over \( j \) gives:

\[
\|\mathbf{u} - \mathbf{v}_I\|^2_E = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} [u - u_I]'^2 \, dx \leq \sum_{j=1}^n (x_j - x_{j-1}) \int_{x_{j-1}}^{x_j} [u''(x)]^2 \, dx \leq C\|u''\|
\]

where \( C = C^2 \), \( C \) being the constant in statement.
Let $e = u - u_0$ then the inequality we want to prove is equal to:

$$
\int_{x_{j-1}}^{x_j} [e'(x)]^2 \, dx \leq c \frac{(x_j - x_{j-1})^2}{(x_j - x_{j-1})^4} \int_{x_{j-1}}^{x_j} [e''(x)]^2 \, dx
$$

(simply because $e''(x) = M'' - M''_0 = M''$ as $u_0 = \text{linear}$)

Then we change variables so that we map element $[x_{j-1}, x_j]$ to $[0, 1]$ (this is a recurrent theme in finite elements!)

$$
x(\tilde{x}) = x_{j-1} + \tilde{x}(x_j - x_{j-1})
$$

new variable $\tilde{x} \in [0, 1]$

$$
\tilde{e}(\tilde{x}) = e(x(\tilde{x})) = e(x_{j-1} + \tilde{x}(x_j - x_{j-1}))
$$

$$
(x_j - x_{j-1}) \int_0^1 [e'(x(\tilde{x}))]^2 \, d\tilde{x} \leq c \frac{(x_j - x_{j-1})^2}{(x_j - x_{j-1})^4} \int_0^1 [e''(x(\tilde{x}))]^2 \, d\tilde{x}
$$

But by the chain rule:

$$
\tilde{e}'(\tilde{x}) = (x_j - x_{j-1}) e'(x(\tilde{x}))
$$

$$
\tilde{e}''(\tilde{x}) = (x_j - x_{j-1})^2 e''(x(\tilde{x}))
$$

Thus:

$$
\int_0^1 [\tilde{e}''(\tilde{x})]^2 \, d\tilde{x} \leq c \int_0^1 [\tilde{e}''(\tilde{x})]^2 \, d\tilde{x}
$$

Now we need to check estimates:

$$
\int_0^1 [\tilde{e}'(\tilde{x})]^2 \, d\tilde{x} \leq c \int_0^1 [\tilde{e}''(\tilde{x})]^2 \, d\tilde{x}
$$

which is an analysis exercise, we shall do this with a different notation for clarity:

$$
\int_0^1 (\tilde{e}'(\tilde{x}))^2 \, d\tilde{x} \leq c \int_0^1 (\tilde{e}''(\tilde{x}))^2 \, d\tilde{x}
$$
First note that \( \omega'(0) = \omega'(1) = 0 \) since the interpolant \( \omega \) coincides with function on all nodes.

\( \Rightarrow \) By Rolle's (or MVT): \( \exists \xi \in (0, 1) \) s.t. \( \omega'() = 0 \)

\[
\frac{\omega'(y) - \omega'(\xi)}{y - \xi} = \int_\xi^y \omega''(x) \, dx
\]

By Schwarz:

\[
|\omega'(y)| = \left| \int_\xi^y \omega''(x) \, dx \right| = \left| \int_\xi^y \omega''(x) \, dx \right|
\]

\[
\leq \sqrt{\int_\xi^y 1 \, dx} \cdot \sqrt{\int_\xi^y (\omega''(x))^2 \, dx} \frac{1}{2} = |y - \xi| \cdot \sqrt{\int_\xi^y (\omega''(x))^2 \, dx} \frac{1}{2}
\]

Squaring and integrating we get result since

\[
\max_{0 \leq \xi \leq 1} \int_0^1 |y - \xi| \, dy = \frac{1}{2}
\]
Key idea: Compute \( K = [a(\phi_i, \phi_j)]_{i,j=1}^n \) by summing the local contributions of each element.

We need a local-to-global index \( i(e,j) \) that relates the local degree of freedom \( j \) on element \( e \) to its global index \( j \).

Recall an 1D example:

\[ x_0 \leq x_1 \leq x_2 \ldots \leq x_n \]

The elements are \( I_e = [x_{e-1}, x_e] \), \( e=1 \ldots n \).
Each element has 2 degrees of freedom: (the node of interval)

\[ I_e \]
\[ j = 0, j+1 \]

and in order to get the global index associated with node \( j \) on element \( e \) we need:

\[ i(e,j) = e + j + 1, \quad e = 1 \ldots n, \quad j = 0, 1. \]

(Obviously this relation depends on particular triangulation you make, and is usually more complicated than this).

We may rewrite the interpolant of a continuous function \( f \) for the space of all piecewise linear fun on grid with:

\[
   f_I = \sum_{i=0}^{n} f(x_i) \phi_i = \sum_{e} \sum_{j=0}^{A} f(x_{i(e,j)}) \phi_j^e
\]

Note: the expression gets values at \( x_i \) wrong (added twice) but this doesn't matter when can plug in integrals!
where \( \phi_j \) are the basis functions for linear functions on the single interval \( I_e = [x_{e-1}, x_e] \)

\[
\phi_j(x) = \phi_j \left( \frac{x - x_{e-1}}{x_e - x_{e-1}} \right)
\]

where

\[
\phi_0(x) = \begin{cases} 1-x & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}
\]

\[
\phi_1(x) = \begin{cases} x & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}
\]

Note: (this will happen in 2D too) We have related the "local" basis fun \( \phi_e \) to those at a reference (parent) element \( [0, 1] \) via an affine mapping \( [0, 1] \rightarrow [x_{e-1}, x_e] \)

\[
(\text{in 2D}) \quad \frac{1}{2} \rightarrow \triangle \text{ arbitrary triangle in triangulation}
\]

Therefore in order to construct bilinear form \( a(u, v) \):

\[
a(u, v) = \sum_e a_e (u, v)
\]

where the "local bilinear form" is:

\[
a_e (u, v) = \int_{I_e} u v \, dx
\]

\[
= \frac{1}{|I_e|} \int_0^1 \left( \sum_{j=0}^1 \bar{u}_j e_i \phi_j \right) \left( \sum_{j=0}^1 \bar{v}_j (e, j) \phi_j \right) \, dx
\]

\[
= \frac{1}{|I_e|} \left[ \bar{v}_i (e, 0) \bar{v}_i (e, 1) \right] K_{bc} \left[ \bar{v}_i (e, 0) \bar{v}_i (e, 1) \right]
\]
where \((K_{el})_{ij} = \int \phi_i^{j-1} \phi_j^{j-1} \text{d}x\)

"local stiffness matrix" for \(i = 1, 2\).

The construction of a finite element space

Recall (WF):

Find \(u \in V\) such that \(a(u, v) = F(v)\) for all \(v \in V\).

We need to construct a finite dimensional \(SCV\). If we look at how we defined such \(SCV\) in D we needed to specify the following:

1. What is the function restricted to an element \(\rightarrow \text{linear}\)
2. How does a function determined in an element \(\rightarrow \text{by its nodal values}\)
3. How do the restrictions of the function to 2 neighboring elements match at the boundary \(\rightarrow \text{same nodal value, continuity}\).

Definition (Finite Element)

A triplet \((K, P, N)\) is called a finite element if:

i) \(K \subset \mathbb{R}^m\) is a bounded closed set with non-empty interior and piece smooth body, ("element domain")

ii) \(P = \text{finite dimensional space of functions on } K\)
    ("shape functions")

iii) \(N = \{N_1, N_2, \ldots, N_k\} = \text{basis of dual of } P\) (d.o.f.)
Some Hilbert space theory

\[ P' = \text{dual of } P = \text{vector space of all linear functionals of } P \]

\[ \dim P' = \dim P \quad (\text{when } P \text{ is finite dim.}) \]

(a linear functional is a linear mapping \( P \rightarrow \mathbb{R} \))

**Definition (modal basis)** Let \((K, P, N)\) be a finite element. The basis \( \{\phi_1, \phi_2, \ldots, \phi_f\} \) of \( P \) s.t.

\[ N_i (\phi_j) = \delta_{ij} \]

is called modal basis of \( P \).

**Example (1D)**: \( K = [0, 1] \), \( P = P_1 \) = linear poly

\[ N = \{N_1, N_2\} \quad \text{with} \quad N_1 (0) = 0 \quad N_2 (1) = 0 \]

\( (K, P, N) \) = finite element

and modal basis is:

\[ \phi_1 (x) = -x \]
\[ \phi_2 (x) = x \]

However other elements are possible; for example \( P_2 \) elements with d.o.f. being values at end points of interval and at center.

\[ 0 \quad \frac{1}{2} \quad 1 \]