THE FINITE ELEMENT METHOD

This is the plan for this part of the class

§11.5 Rayleigh-Ritz method (FEM for BVP) } Brenner & Scott
§12.4 The FEM for elliptic problems

§12.1 - 12.3 Elliptic, parabolic and hyperbolic problems } Burden & Faires, Kincaid & Cheney

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What is the finite element method? It's a systematic way of generating discrete algorithms for approximating the solution of differential equations.

We start first with an overview of FEM by looking at BVP:

$$(BVP) \begin{cases} -\dfrac{d^2u}{dx^2} = f & \text{in } (0,1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

If u is the solution and v is any suff. smooth function with $v(0)=0$:

$$(f,v) := \int_0^1 f(x)\,v(x)\,dx = -\int_0^1 u''v\,dx \underset{IBP}{=} \underbrace{-u'v\Big|_0^1}_{=0} + \underbrace{\int_0^1 u'v'\,dx}_{:=a(u,v)}$$

Thus we reformulate (BVP) as follows:

Let $V = \{ v \in L^2(0,1) \mid a(v,v) < \infty \text{ and } v(0)=0 \}$

If u solves (BVP) then $u \in V$ and

$$\boxed{\forall v \in V \quad a(u,v) = (f,v)} \qquad (WF)$$

This is the "variational" or "weak" formulation of BVP.

Can we show that if u satisfies (WF) then u solves (BVP)?

This holds if we assume additional regularity of f and u:

<u>Theorem</u>: Let $f \in C^0([0,1])$ and $u \in C^2([0,1])$ satisfy (WF). Then u solves (BVP).

<u>Proof</u>: Let $v \in V \cap C^1([0,1])$.

$$(*) \quad (f,v) = a(u,v) = \int_0^1 u'v'\,dx \underset{\text{IBP}}{=} \underbrace{u'v\Big|_0^1}_{=u'(1)v(1)} - \int_0^1 u''v\,dx.$$

Choosing $v \in V$ s.t. $v(1)=0$:

$$(f,v) = -\int_0^1 u''v\,dx$$

$$\Leftrightarrow \quad (\underbrace{f+u''}_{w \in C^0}, v) = 0.$$

Assume for contradiction that $w \neq 0$. Then w must assume one sign on some interval $[a,b] \subset [0,1]$ (by continuity). Choose

$$v(x) = \begin{cases} (x-a)^2(x-b)^2 & \text{in } [a,b] \\ 0 & \text{otherwise} \end{cases}$$

(or any other v with one sign for that matter)

then $(w,v) \neq 0$ which is a contradiction. $\Rightarrow -u'' = f$.

Now for B.C.:

use $v(x) = x$ in $(*)$ to get:

$$u'(1) = (f+u'', v) = 0$$

The other B.C. comes from $u \in V$:

$$u(0) = 0$$

QED.

The following nomenclature is given to boundary conditions.

$u(x)=0$ = <u>Dirichlet</u> B.C. $\equiv$ essential B.C. because it appears explicitly in definition of V.

$u'(x)=0$ = <u>Neumann</u> B.C. = natural B.C. because it appears implicitly in variational formulation

In light of previous theorem it is clear that the weak formulation is a way of interpreting (BVP) with more loose assumptions on f.

Ritz Galerkin Approximation

Let $S \subset V$ be any finite dimensional subspace of V.
Replacing V by S in (WF) we get:

$$\boxed{\text{Find } u_S \in S \text{ such that } a(u_S, v) = (f, v) \quad \forall v \in S} \qquad (RG)$$

We will see that by this simple device we have created a linear system from BVP!

<u>Theorem</u> Given $f \in L^2(0,1)$, (RG) has a unique solution.

<u>proof</u>: Let $\{\phi_i\}_{i=1}^{n}$ be a basis of S. We rewrite (RG) in terms of this basis:

Let $u_S = \sum_{i=1}^{n} U_i \phi_i$ then:

$$a(u_S, v) = \sum_{i=1}^{n} U_i\, a(\phi_i, v) = (f, v) \quad \forall v \in S$$

$$\Leftrightarrow \quad a(u_S, \phi_j) = \sum_{i=1}^{n} U_i\, a(\phi_i, \phi_j) = (f, \phi_j) \quad j = 1 \dots n$$

$$\Leftrightarrow \quad KU = F$$

where $K \in \mathbb{R}^{n \times n}$ and $K_{ij} = a(\phi_j, \phi_i)$

$F \in \mathbb{R}^n$, $\quad F_i = (f, \phi_i)$

$U \in \mathbb{R}^n$ $\quad U = \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix}$

This linear system has a unique solution: if it did not then $\exists\ V \neq 0$ s.t. $KV = 0$

Let $v = \sum_{j=1}^{n} V_j \phi_j$ then

$$a(v, \phi_j) = 0 \quad j = 1..n$$

$$\Rightarrow \quad \sum_{j=1}^{n} V_j a(v, \phi_j) = 0$$

$$\Leftrightarrow \quad a(v, v) = 0 = \int_0^1 (v')^2\, dx$$

$$\Rightarrow \quad v' = 0 \Rightarrow v = \text{constant} = v(0) = 0.$$

$$\Rightarrow \quad V = 0. \qquad \text{QED.}$$

<u>note</u>: some technical points have been swept under the rug here.

K = stiffness matrix

$K = K^T$ because the "energy inner prod" $a(\cdot,\cdot)$ is symm (symm)

K is also positive definite since

$$V^T K V = \sum_{i,j=1}^{n} k_{ij} v_i v_j = a(v, v) \geq 0 \quad \text{for } v = \sum_{j=1}^{n} v_j \phi_j$$

and $V^T K V = 0 \Rightarrow v = 0 \Rightarrow V = 0.$

Error Estimates (is the approx any good?)

Galerkin orthogonality:

$$a(u, w) = (f, w) \quad \forall w \in S$$

$$a(u_S, w) = (f, w)$$

$$\Rightarrow \boxed{a(u - u_S, w) = 0 \ \forall w \in S}$$

Now introduce the so called energy norm:

$\|u\|_E = (a(u,u))^{\frac{1}{2}}$

which satisfies Schwartz inequality:

$$|a(v, w)| \leq \|v\|_E \|w\|_E \quad \forall v, w \in V$$

Then $\forall v \in S$:

$$\|u - u_S\|_E^2 = a(u - u_S, u - u_S)$$

$$= a(u - u_S, u - v) + \underbrace{a(u - u_S, v - u_S)}_{=0 \text{ by Galerkin } \perp}$$

$$\leq \|u - u_S\|_E \|u - v\|_E$$

If $\|u - u_S\|_E \neq 0$ then

$$\|u - u_S\|_E \leq \|u - v\|_E \quad \forall v \in S$$

$$\Rightarrow \quad \|u - u_S\|_E \leq \inf_{v \in S} \|u - v\|_E$$

Since $u_S \in S$:

$$\inf_{v \in S} \|u - v\|_E \leq \|u - u_S\|_E$$

Therefore:

$$\|u - u_S\|_E = \inf_{v \in S} \|u - v\|_E$$

Moreover there is an element (namely u_S) in S for which the infimum is attained. Therefore we get: (79)

$$\boxed{\textbf{Theorem} \quad \|u - u_S\|_E = \min_{v \in S} \|u - v\|.}$$

What this says is that the RG approx gives automatically the best approx. to u in S, as measured by the energy norm.

Now let's see what happens with the L^2 norm:

$$\|v\| = (v, v)^{\frac{1}{2}} = \left(\int_0^1 v^2 dx \right)^{\frac{1}{2}}$$

To estimate $\|u - u_S\|$ we use a <u>duality</u> <u>argument</u> (Aubin-Nitsche trick)

Let w be the solution of:

$$\begin{cases} -w'' = u - u_S & \text{on } [0,1] \\ w(0) = w'(1) = 0 \end{cases}$$

Then:

$$\begin{aligned}
\|u - u_S\|^2 &= (u - u_S, u - u_S) \\
&= (u - u_S, -w'') \\
&\underset{\text{IBP}}{=} a(u - u_S, w) - \underbrace{\cancel{(u - u_S) w' \Big|_0^1}}_{=0} \\
&\underset{\text{Galerkin}\perp}{=} a(u - u_S, w - v) \quad \forall v \in S
\end{aligned}$$

Thus using Schwarz inequality

$$\|u - u_S\|^2 \leq \|u - u_S\|_E \|w - v\|_E \qquad \|w''\|$$

$$\|u - u_S\| \leq \|u - u_S\|_E \|w - v\|_E / \overbrace{\|u - u_S\|}$$

Taking the infimum over $v \in S$ we get:

$$\|u - u_S\| \leq \|u - u_S\|_E \inf_{v \in S} \frac{\|w - v\|_E}{\|w''\|}$$

Thus the L^2 error can be made much smaller than the energy norm error provided we make the assumption that for some $\varepsilon > 0$:

$$\inf_{v \in S} \|w - v\|_E \leq \varepsilon \|w''\|$$

This is an "approximation assumption" and we get:

$$\|u - u_S\| \leq \varepsilon \|u - u_S\|_E \underset{\substack{\uparrow \\ \text{approx. assump.} \\ \text{again with } w \\ \text{replaced by } u}}{\leq} \varepsilon \left(\varepsilon \|u''\| \right) = \varepsilon^2 \|u''\| = \varepsilon^2 \|f\|.$$

To summarize: $\|u - u_S\|_E \leq \varepsilon \|f\|$

$\|u - u_S\| \leq \varepsilon^2 \|f\|$ -> smaller error

We shall construct subspace (finite elements) where ε can be made arbitrarily small.