11. BOUNDARY VALUE PROBLEMS

We will see methods to solve boundary value problems (BVP) of the kind:

$$(*)\quad \begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha \\ y(b) = \beta \end{cases}$$

Here is one existence theorem due to Keller

__Theorem__ Suppose the function f in $(*)$ is continuous on

$$D = \{(x, y, y') \mid x \in [a, b],\ y \in \mathbb{R},\ y' \in \mathbb{R}\}$$

and also f_y and $f_{y'}$. If

i) $f_y(x, y, y') > 0 \quad \forall (x, y, y') \in D$

ii) $\exists M > 0$ s.t.

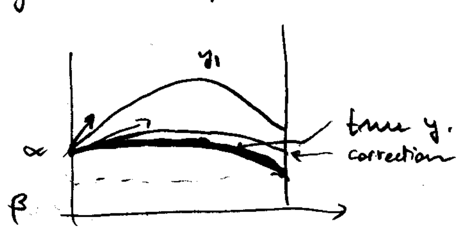
$$|f_{y'}(x, y, y')| \leq M \quad \forall \quad (x, y, y') \in D$$

then BVP $(*)$ has a <u>unique</u> <u>solution</u>.

This is of course only a sufficient condition for uniqueness, and is not by any means a precondition for the methods we shall see to work.

§ 11.1-2: Shooting methods

This class of methods solves a related initial-value problem with a <u>guess</u> for $y'(a)$, hoping that $y(b) = \beta$. If it is not, the guess for $y'(a)$ is improved (how?) the IVP solved again and so on:



So the kind of IVP we solve is:

$$\begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha \,;\ y'(a) = \zeta \end{cases}$$ which has solution y_ζ

The goal is to choose ζ s.t. $y_\zeta(b) = \beta$. Let us call the mistake:

$$\boxed{\phi(\zeta) = y_\zeta(b) - \beta}.$$

Shooting methods are essentially trying to find a root of $\phi(\zeta)$, so we could conceivably use:

- bisection method
- secant method
- Newton's method ...

however we have to be careful with the number of function evaluations, because every evaluation of ϕ requires a solve of IVP between a and b.

For the particular case of linear BVP we can find the root of ϕ directly.

Def (linear BVP):

$$(L)\ \begin{cases} y'' = p(x)\, y' + q(x)\, y + r(x), & x \in [a, b] \\ y(a) = \alpha \,;\ y(b) = \beta \end{cases}$$

Corollary of existence theorem: Problem (L) has a unique sol. if p, q, r are continuous on $[a, b]$, and $q(x) > 0$ on $[a, b]$.

For linear BVP $\phi(\zeta)$ is linear too, so it's enough to solve two IVP for say ζ_1 and ζ_2 to find the root.

Let y_1 be the sol of IVP associated with (L) $y_1(a) = \alpha$; $y_1'(a) = \xi_1$

y_2 ——— $y_2(a) = \alpha$; $y_2'(b) = \xi_2$

Then $y(t) = \lambda y_1(t) + (1-\lambda) y_2(t)$ satisfies DE and

$$y(a) = \lambda y_1(a) + (1-\lambda) y_2(a) = \alpha$$
$$y(b) = \lambda y_1(b) + (1-\lambda) y_2(b) = \beta$$

$$\Rightarrow \boxed{\lambda = \frac{\beta - y_2(b)}{y_1(b) - y_2(b)}}$$

Thus y with this particular choice of λ solves (L).

So the <u>linear shooting method</u> is:

- solve DE of (L) + init. cond. $y_1(a) = \alpha$; $y_1'(a) = 0$. Keep iterates in a vector $\underline{y_1}$.
- solve DE of (L) + init. cond $y_2(a) = \alpha$; $y_2'(a) = 1$. keep iterates in a vector $\underline{y_2}$

Can be grouped as one first order system solve see below

- compute λ
- output $\underline{y} = \underline{y_1} + \underline{y_2}$.

Of course the methods we use for solving IVP have to use same time steps for this approach to make sense (so no adaptivity!)

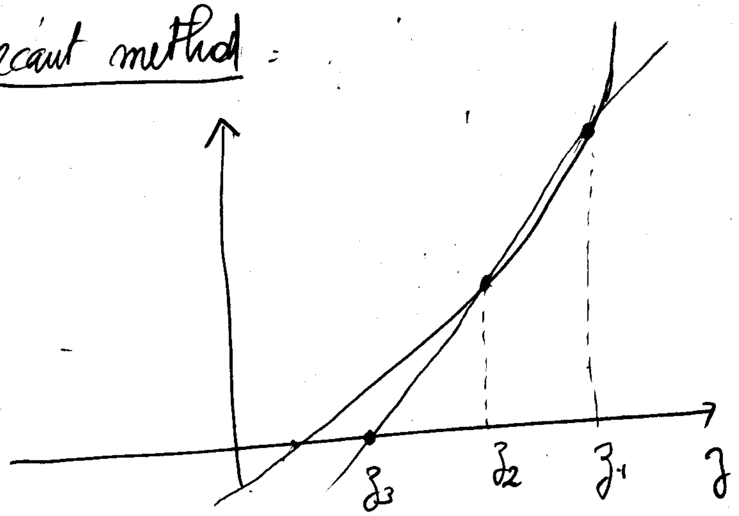
The two first steps of the linear shooting method can be solved using a solver for systems of eq:

$$\begin{cases} y_1' = y_3 \\ y_2' = y_4 \\ y_3' = f(t, y_1, y_3) \\ y_4' = f(t, y_2, y_4) \end{cases} \iff \underline{y}' = \underline{f}(t, \underline{y}) \quad + \text{I.C.}$$

$y_1(a) = \alpha$; $y_2(a) = \alpha$; $y_3(a) = 0$; $y_4(a) = 1$.

What about the non-linear problem?

Secant method:



If $\phi(z)$ were linear then:

$$\phi(z) - \phi(z_2) = \left[\frac{\phi(z_1)-\phi(z_2)}{z_1 - z_2}\right](z - z_2)$$

Now we solve for z_3 s.t. $\phi(z_3) = 0$:

$$\underbrace{\cancel{\phi(z_3)}}_{0} - \phi(z_2) = \left[\frac{\phi(z_1)-\phi(z_2)}{z_1 - z_2}\right](z_3 - z_2)$$

$$\Rightarrow \quad z_3 = z_2 - \frac{z_2 - z_1}{\phi(z_2) - \phi(z_1)}\,\phi(z_2)$$

or more generally:

$$\boxed{z_{n+1} = z_n - \frac{z_n - z_{n-1}}{\phi(z_n) - \phi(z_{n-1})}\,\phi(z_{n-1})}$$

We know from root-finding that Newton's method is faster than the secant method. What would we require?

The Newton's iteration to find zero of $\phi(z) = y_z(b) - \beta$ is:

$$\boxed{z_{n+1} = z_n - \frac{\phi(z_n)}{\phi'(z_n)}}$$

where y_z solves:

$$(PZ) \quad \begin{cases} y_z'' = f(t, y_z, y_z') \\ y_z(a) = \alpha \,;\ y_z'(a) = z \end{cases}$$

Here is a trick to obtain ϕ': differentiate (PZ) wrt z:

$$\begin{cases} \dfrac{\partial y_z''}{\partial z} = \dfrac{\partial t}{\partial z} \dfrac{\partial f}{\partial t} + \dfrac{\partial y_z}{\partial z} \dfrac{\partial f}{\partial y_z} + \dfrac{\partial y_z'}{\partial z} \dfrac{\partial f}{\partial y_z'} \\ \dfrac{\partial}{\partial z} y_z(a) = 0 \,;\ \dfrac{\partial}{\partial z} y_z'(a) = 1 \end{cases}$$

If we let $v = \dfrac{\partial y_z}{\partial z}$ we get the "first variational" eq:

$$\begin{cases} v'' = f_{y_z}(t, y_z, y_z')\, v + f_{y_z'}(t, y_z, y_z')\, v' \\ v(a) = 0, \quad v'(a) = 1 \end{cases}$$

which can be solved at the same time as (PZ) using Chap 5.

$$v(b) = \frac{\partial y_z}{\partial z}(b) = \phi'(z)$$

Thus the iteration,

$$\boxed{z_{n+1} = z_n - \frac{\phi(z_n)}{v(b)}}$$

§ 11.3 - 11.4 Finite Difference methods

$$(BVP)\begin{cases} y'' = f(t,y,y') \\ y(a)=\alpha;\ y(b)=\beta \end{cases}$$

Idea: • Subdivide $[a,b]$ in N+1 subintervals of length $\frac{b-a}{N+1} \equiv h$

$a=t_0 \quad t_1 \quad t_2 \ldots \quad t_N \quad b=t_{N+1}$, $t_i = a+ih$

• Replace y'', y' by discrete approx (finite differences).

• If BVP is linear i.e. $f(t,y,y') = py'+qy+r$ form a system and solve.

Otherwise use Newton-type approach.

Finite differences:

Taylor's theorem:

$$y(t_{i+1}) = y(t_i+h) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(t_i) + \frac{h^4}{24}y^{(4)}(\xi_i^+) \quad \xi_i^+ \in [t_i, t_i+h]$$

$$y(t_{i-1}) = y(t_i-h) = y(t_i) - hy'(t_i) + \frac{h^2}{2}y''(t_i) - \frac{h^3}{6}y'''(t_i) + \frac{h^4}{24}y^{(4)}(\xi_i^-) \quad \xi_i^- \in [t_{i-1}, t_i]$$

$+$

$$\frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} = y''(t_i) + \frac{h^2}{24}\underbrace{\left(y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-)\right)}_{= 2y^{(4)}(\xi_i)}$$

$\xi_i \in (t_{i-1}, t_{i+1})$ Intermediate Value Theo.

$$\Rightarrow \boxed{y''(t_i) = \frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} + \mathcal{O}(h^2)}$$

Similar one can show:

$$\boxed{y'(t_i) = \frac{y(t_{i+1}) - y(t_{i-1})}{2h} + \mathcal{O}(h^2)}$$

So the discrete version of BVP is:

$$\begin{cases} y_0 = \alpha \\ \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) \\ y_{n+1} = \beta \end{cases}$$

Linear case:

When f is linear we get a system:

$$\begin{cases} y_0 = \alpha \\ \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = p_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + q_i y_i + r_i, \\ y_{N+1} = \beta \end{cases}$$

$\iff \quad A\underline{y} = \underline{b}$

where $A = L - D - Q \in \mathbb{R}^{n\times n}$

$$= \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} - \frac{1}{2h}\begin{bmatrix} 0 & -p_1 & & & \\ p_2 & 0 & -p_2 & & \\ & p_3 & 0 & -p_3 & \\ & & \ddots & \ddots & -p_{n-1} \\ & & & p_n & 0 \end{bmatrix} - \begin{bmatrix} q_1 & & & \\ & q_2 & & \\ & & \ddots & \\ & & & q_n \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} r_1 - \alpha/h^2 - \frac{p_1\alpha}{2h} \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n - \beta/h^2 + \frac{p_n\beta}{2h} \end{bmatrix}$$

A = tridiagonal cheap solve.

Theorem: Suppose p, q, r are continuous on $[a,b]$. If $q(t) \geq 0$ on $[a,b]$ then system $A\underline{y} = \underline{b}$ has a unique sol provided:

$$h < \frac{2}{M}, \text{ where } M = \max_{x\in[a,b]} |p(t)|.$$

Accuracy one can expect is $O(h^2)$

<u>Non Linear case</u>: $\begin{cases} y' = f(t,y,y') \\ y(a)=\alpha;\ y(b)=\beta \end{cases}$

We assume:

- $f,\ f_y,\ f_{y'}$ continuous on $D = \{(t,y,y') \mid t\in[a,b],\ y\in\mathbb{R},\ y'\in\mathbb{R}\}$
- $f_y(t,y,y') \geq \delta$ on D, for $\delta > 0$.
- $\exists k, L > 0$ s.t.
  $k = \max_{(t,y,y')\in D} |f_y(t,y,y')|$; $L = \max_{(t,y,y')\in D} |f_{y'}(t,y,y')|$

$\rightsquigarrow$ guarantees existence of a unique sol.

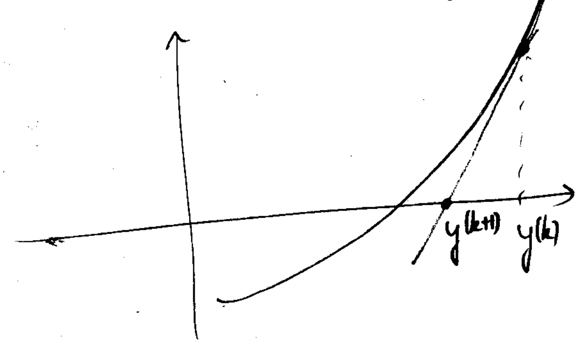
Using finite difference formula we get to:

$$\begin{cases} \dfrac{\alpha - 2y_1 + y_2}{h^2} - f\left(t_1, y_1, \dfrac{y_2-\alpha}{2h}\right) \equiv F_1(\underline{y}) = 0 \\[2ex] \dfrac{y_1 - 2y_2 + y_3}{h^2} - f\left(t_2, y_2, \dfrac{y_3 - y_1}{2h}\right) \equiv F_2(\underline{y}) = 0 \\ \vdots \\ \dfrac{y_n - 2y_2 + \beta}{h^2} - h\left(t_n, y_n, \dfrac{\beta - y_{n-1}}{2h}\right) \equiv F_3(\underline{y}) = 0 \end{cases}$$

$\Leftrightarrow \quad F(\underline{y}) = 0$ ($n\times n$ nonlinear system of eq., which has unique sol provided $h < \frac{2}{L}$)

Solution using <u>Newton's method</u>:

$$DF(\underline{y}^{(k)})(\underline{y}^{(k+1)} - \underline{y}^{(k)}) = -F(\underline{y}^{(k)})$$



$DF(\underline{y})$ = Jacobian

$$= \begin{bmatrix} \nabla F_1^T \\ \vdots \\ \nabla F_n^T \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial F_n}{\partial y_1} & \cdots & \frac{\partial F_n}{\partial y_n} \end{bmatrix}$$

For $k=0 \dots$ until convergence

$$\underline{y}^{(k+1)} = \underline{y}^{(k)} - DF(\underline{y}^{(k)})^{-1} F(y^{(k)})$$

- Typical $\underline{y}^{(0)}$ has entries: $y_i = \alpha + i\left(\frac{\beta-\alpha}{b-a}\right)h$ and satisfies B.C.

- $$DF(y) = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & -2 & 1 \\ & & 1 & -2 \end{bmatrix}$$

$$- \begin{bmatrix} f_y(t_1, y_1, \frac{y_2-\alpha}{2h}) & \frac{1}{2h} f_{y'}(t_1, y_1, \frac{y_2-\alpha}{2h}) & & \\ -\frac{1}{2h} f_{y'}(t_2, y_2, \frac{y_3-y_1}{2h}) & f_y(t_2, y_2, \frac{y_3-y_1}{2h}) & \frac{1}{2h} f_y(t_2, y_2, y_3-y_1) & \\ & \ddots & \ddots & \ddots \\ & & -\frac{1}{2h} f_y(t_n, y_n, \frac{\beta-y_{n-1}}{2h}) & f_y(t_n, y_n, \frac{\beta-y_{n-1}}{2h}) \end{bmatrix}$$

= tridiagonal matrix, so cheap to solve