§5.10 Stability (this comes from K&C. B&F is more general)

Recall the general form of a multistep method:

\[(\star) \quad ak \cdot y_n + a_{k-1} \cdot y_{n-1} + \ldots + a_0 \cdot y_{n-k} = h \left[ bk \cdot f_n + b_{k-1} \cdot f_{n-1} + \ldots + b_0 \cdot f_{n-k} \right] \]

where \( f_n = f(t_n, y_n) \).

\[ bk \neq 0 \Rightarrow \text{implicit method (new } y_n \text{ appear on both sides)} \]
\[ bk = 0 \Rightarrow \text{explicit method} \]

We associate two polynomials with (\star):

\[ p(z) = a_k z^k + a_{k-1} z^{k-1} + \ldots + a_0 \]
\[ q(z) = b_k z^k + b_{k-1} z^{k-1} + \ldots + b_0 \]

**Def (convergent method):** Let \( y_h(t) \) be the approx sol obtained by using a numerical method with step size \( h \). The method is said to be convergent if:

\[ \forall t \in [t_0, t_m], \lim_{h \to 0} y_h(ht) = y(t) \]

provided the starting values obey same eq, i.e.:

for all \( n \) s.t. \( 0 \leq n \leq k-1 \):

\[ \lim_{h \to 0} y_h(h, t_0 + nh) = y(t_0 + nh) \]

and \( f \) satisfies conditions for the problem \( y' = f \) to be well posed.
Def: (Stability) A multistep method is said to be **stable**

if all the roots of polynomial lie in the disk $|z| < 1$ and

if each root $|z| = 1$ is **simple** ($\neq$ multiplicity 1)

(Strongly) stable

only root with $|z| = 1$

$z = 1$

(Weakly) stable

more than one root

with $|z| = 1$

Unstable

$p(z) = (z-1)^2 (z + 1)$

Def: (Consistency) A multistep method is said to be **consistent** of:

$p(1) = 0$

and $p'(1) = q(1)$

(called will see in a moment, where this comes from)

Theorem: For multistep methods of general form (*):

$$\text{Convergent } \iff \text{ (stable and consistent) }$$

**Proof:** Stable and consistent $\implies$ Convergent is very involved.

**Convergent $\implies$ Stable** (stability is a necessary condition for convergence)

Assume method is not stable, we will give a simple problem where method is not convergent.
method not stable $\Rightarrow \exists$ root $\lambda$ of $p(\lambda)$ with $|\lambda| > 1$

or $\exists$ \________ with $|\lambda| = 1$ and $p'(\lambda) = 0$

\(\text{not: } p(\lambda) = 0 \Rightarrow p(\lambda) = (\lambda - \gamma) r(\lambda) p'(\gamma) = r(\lambda) + (\gamma - \lambda) r'(\lambda) p'(\gamma) = r(\lambda)\)

thus $p'(\lambda) = 0 \Rightarrow \lambda$ is a \underline{multiple} root of $p$.

Consider the simple IVP:
\[
\begin{align*}
\text{exact sol to } y'(t) = 0 \\
y(0) = 0
\end{align*}
\]

Applying (*):
\[
ak y_n + ak-1 y_{n-1} + \ldots + a_0 y_{n-k} = 0 \quad (4)
\]

This is a \underline{difference eq} and it is relatively easy to come up with \underline{sequence} satisfying it (see below for a refresher on difference eq.). In particular any sequence of the form:

\[y_n = \lambda^n, \quad \lambda \text{ root of } p\]

\underline{satisfies} the difference eq.

\(\text{If } |\lambda| > 1:\)
\[
|y(h, nh)| = h |\lambda|^n < h |\lambda|^k \quad \text{for } 0 < n < k-1
\]

thus $|y(h, nh)| \to 0$ (method is convergent in first few steps)

however, if we let $t = nh$ ($oh = t/m$):
\[
|y(h, t)| = |y(h, nh)| = h |\lambda|^n = \frac{t |\lambda|^n}{m} \to \infty \text{ as } m \to \infty
\]

(method blows up for such a simple problem!)
If \(|\lambda| = 1\) and \(p(1) = 0\), a solution to difference eq is:
\[ y_n = \lambda^n n^m \]

method is convergent for first few steps since:
\[ |y(h, m^n)| = \frac{h |y(1)|^n}{m^n} = \frac{h n < k^n}{\frac{1}{m^n}} \quad \text{(for } 0 < m < k \text{)} \]
\[ \downarrow \quad \text{as } k \to 0. \]

method does not converge after a few steps \((t = mh, h = t/m)\):
\[ |y(h, t)| = \frac{h m^{|y|} t^n}{m^n} = t \neq 0 \quad \text{as } k \to 0. \]

\[ \text{Convergent } \Rightarrow \text{ consistent } \]

Assume method (*) is convergent.

\[ \left\{ \begin{array}{c}
 y' = 0 \\
y(0) = 1
\end{array} \right. \quad \text{same difference eq: } a_k y_k + a_{k-1} y_{k-1} + \cdots + a_0 y_0 = 0 \quad (1) \]

\[ \begin{align*}
 (P2) & \quad \text{a sol to (1) so set } y_0 = y_1 = \cdots = y_{k-1} = 1 \\
 & \quad \text{and use (1) to find } y_m, m \geq k. \\
 & \quad \text{Since method is convergent:} \\
 & \quad \lim_{m \to \infty} y_m = 1, \text{ plugging into (1):} \\
 & \quad \Rightarrow \quad a_k + a_{k-1} + \cdots + a_0 = 0 \\
 & \quad \Rightarrow \quad p(1) = 0
\end{align*} \]

We now consider the problem
\[ \left\{ \begin{array}{c}
 y' = 1 \\
y(0) = 0
\end{array} \right. \quad \text{(sol is } y(t) = t) \]

we get a new eq:
\[ a_k y_k + a_{k-1} y_{k-1} + \cdots + a_0 y_{k-0} = h [b_k + b_{k-1} + \cdots + b_0] \quad (2) \]

\[ \text{Convergent } \Rightarrow \text{ stable } \Rightarrow \quad p(1) = 0 \quad (1 \text{ is a simple root}) \]
\[ p'(1) \neq 0 \]
A solution to (2) is given by:

\[ Y_m = (m+k)h \gamma, \quad \text{where} \quad \gamma = \frac{q(1)}{\rho'(1)}. \]

Checking by substitution in LHS of (2):

\[ k \gamma (ak (m+k) + ak-1 (m+k-1) + \ldots + a_0 m) \]
\[ = \alpha h \gamma (ak + ak-1 + \ldots + a_0) + h \gamma \left[ k ak + (k-1) ak-1 + \ldots + a_1 \right] \]
\[ = \rho(1) = 0 \]
\[ = \rho'(1) \]
\[ = \rho(1) = k \left[ b_k + b_{k-1} + \ldots + b_0 \right]. \]

Now the first few steps are consistent with initial value: \( y(0) = 0 \):

\[ |y(\alpha, nh)| = (m+k)h \gamma \to 0 \quad \text{as} \quad h \to 0. \]

(since \( 0 < m < k-1 \))

Since the method is convergent we must have:

\[ \lim_{m \to \infty} Y_m = t \quad \text{when} \quad nh = t \]
\[ \implies \lim_{n \to \infty} (m+k)h \gamma = \lim_{n \to \infty} \frac{t}{n} = 0 \quad \Rightarrow \quad \gamma = 1 \]

\[ \quad \Rightarrow \quad \left( \rho'(1) = \rho(1) \right) \]

Example: Milne's method \( Y_n - Y_{n-2} = h \left[ \frac{1}{3} f_n + \frac{3}{4} f_{n-1} + \frac{1}{3} f_{n-2} \right] \)

\[ \rho(3) = 8^2 - 1 \quad \text{rate:} +4, -1 \quad \text{(ample)} \Rightarrow \text{stable} \]
\[ q(3) = \frac{1}{3} 8^2 + \frac{3}{4} 8 + \frac{1}{3} \]
\[ \rho'(3) = 2 \]
\[ q(1) = 2 = \rho'(1) \]
\[ \rho(4) = 0 \]

sufficient consistent 

method is convergent
Theorem (Complete Solution - Special Case): If \( p(x) \) is a poly with multiplicity \( k \), associate each root \( s \) of \( p(x) \) with \( k \) of its multiplicity. Then a basis for \( \text{nullsp}(p(x)) \) is:

\[
x(a), x(a)^2, x(a)^3, \ldots, x(a)^{k-1}
\]

where \( x(a) = \alpha \) is a root. For each root \( a \) of \( p(x) \), if all \( \alpha \) are simple roots, then a basis for \( \text{nullsp}(p(x)) \) is:

\[
x(a), x(a)^2, x(a)^3, \ldots, x(a)^{k-1}, \ldots
\]

where \( x(a) = \alpha \) is a root of \( p(x) \).

It is not hard to see that:

\[
p(E) = (E - k_1)(E - k_2)(E - k_3) \ldots\]

A linear differential equation:

\[
(E^n x)_{n-1} = 2x
\]

A linear diophantine equation:

\[
x = (2, 3, 3, \ldots)
\]

where \( x = (x_1, x_2, x_3, \ldots) \) is a sequence.

A linear Diophantine equation can be solved using the shift operator.

\[
\alpha = (x_1, x_2, x_3, \ldots)
\]

where \( \alpha = (x_1, x_2, x_3, \ldots) \) is a sequence.