§5.10 Stability (this comes from K&C. B&F is more general)

Recall the general form of a multistep method:

$$(*) \quad a_k y_n + a_{k-1} y_{n-1} + \cdots + a_0 y_{n-k} = h\left[b_k f_n + b_{k-1} f_{n-1} + \cdots + b_0 f_{n-k}\right]$$

where $f_n = f(t_n, y_n)$.

$b_k \neq 0 \Rightarrow$ implicit method (new y_n appears on both sides)

$b_k = 0 \Rightarrow$ explicit method

We associate two polynomials with $(*)$:

$$p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0$$

$$q(z) = b_k z^k + b_{k-1} z^{k-1} + \cdots + b_0$$

Def (Convergent method): Let $y(h,t)$ be the approx sol obtained by using a numerical method with step size h. The method is said to be convergent if:

$\forall t \in [t_0, t_m]$:

$$\lim_{h \to 0} y(h,t) = y(t)$$

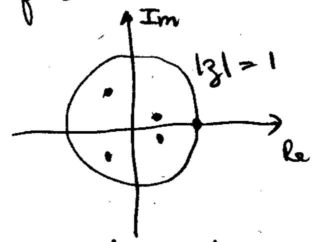
provided the starting values obey same eq, ie:

for all n s.t. $0 \leq n \leq k-1$:

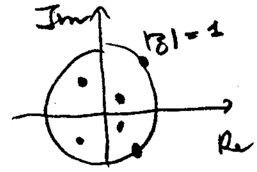
$$\lim_{h \to 0} y(h, t_0 + nh) = y(t_0 + nh).$$

and f satisfies conditions for the problem $\begin{cases} y' = f \\ y(t_0) = \alpha \end{cases}$ to be well posed

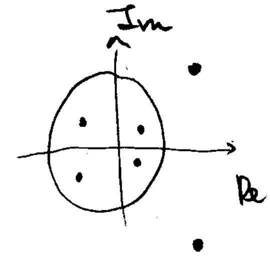
<u>Def (Stability)</u> A multistep method is said to be <u>stable</u> if all the roots of $p(z)$ lie in the disk $|z| \leq 1$ and if <u>each</u> root $|z| = 1$ is <u>simple</u> (= multiplicity 1)



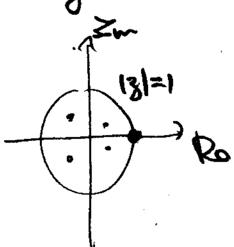
<u>(strongly) stable</u>
only root with $|z| = 1$ is $z = 1$.



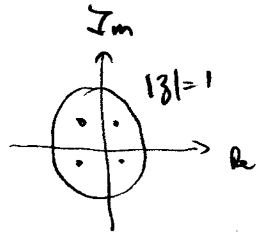
<u>(weakly) stable</u>
more than one root with $|z| = 1$.



<u>unstable</u>



<u>unstable</u>
$p(z) = (z-1)^2 r(z)$



<u>stable</u>
All roots λ, $|\lambda| < 1$

etc...

<u>Def (consistency)</u> A multistep method is said to be <u>consistent</u> if:

$$p(1) = 0$$

and $p'(1) = q(1)$ (we will see in a moment where this comes from)

<u>Theorem</u> For multistep methods of general form (*):

$$\boxed{\text{convergent} \Leftrightarrow (\text{stable and consistent})}$$

<u>proof</u>: <u>stable and consistent $\Rightarrow$ convergent</u> is very involved.

- $\boxed{\text{convergent} \Rightarrow \text{stable}}$ (stability is a necessary cdt° for convergence)

Assume method is not stable, we will give a simple problem where method is not convergent.

method not stable $\Rightarrow$ Ⓐ $\exists$ root λ of $p(z)$ with $|\lambda|>1$

or Ⓑ $\exists$ ———————— with $|\lambda|=1$ and $p'(\lambda)=0$

(note: $p(\lambda)=0 \Rightarrow p(z)=(z-\lambda)r(z)$

$$p'(z)=r(z)+(z-\lambda)r'(z)$$
$$p'(\lambda)=r(\lambda).$$

thus $p'(\lambda)=0 \Leftrightarrow \lambda$ is a multiple root of p)

Consider the simple IVP:

$$(P1)\begin{cases} y'=0 \\ y(0)=0 \end{cases} \qquad (\text{exact sol is } y(t)=0)$$

Applying $(*)$:

$$a_k y_n + a_{k-1} y_{n-1} + \dots + a_0 y_{n-k} = 0 \qquad (1)$$

This is a difference eq and it is relatively easy to come up with sequences satisfying it (see below for a refresher on difference eq.). In particular any sequence of the form:

$$y_n = h\lambda^n, \quad \lambda \text{ root of } p.$$

satisfies the difference eq.

Ⓐ If $|\lambda|>1$:

$$|y(h,nh)| = h|\lambda|^n < h|\lambda|^k \quad \text{for } 0 \le n \le k-1$$

thus $|y(h,nh)| \to 0$ (method is convergent in first few steps)

however, if we let $t=nh$ (or $h=t/n$):

$$|y(h,t)| = |y(h,nh)| = h|\lambda|^n = \frac{t}{n}|\lambda|^n \to \infty \text{ as } n\to\infty$$

(method blows up for such a simple problem!)

(B) $\underline{\text{if } |\lambda| = 1 \text{ and } p(\lambda) = 0}$ a sol to difference eq is:

$y_n = h n \lambda^n$

method is convergent for first few steps since:

$$|y(h, nh)| = h n \underbrace{|\lambda|^n}_{=1} = h n < \underset{\substack{\downarrow \\ 0 \text{ as } h \to 0.}}{hk} \quad (\text{for } 0 \le n \le k-1)$$

method does not converge after a few steps ($t = nh$, $h = t/n$):

$$|y(h, \underset{nh}{\overset{\shortparallel}{t}})| = \underbrace{h n}_{=t} |\lambda|^n = t \ne 0 \quad \text{as } h \to 0.$$

- $\boxed{\text{Convergent} \Rightarrow \text{consistent}}$

Assume method (*) is convergent.

$$(P2) \begin{cases} y' = 0 \\ y(0) = 1 \end{cases}$$

$\rightsquigarrow$ same difference eq $a_k y_n + a_{k-1} y_{n-1} + \dots + a_0 y_{n-k} = 0$ (1)

a sol to (1) is to set $y_0 = y_1 = \dots = y_{k-1} = 1$

and use (1) to find y_n, $n \geq k$.

Since method is convergent:

$\lim_{n \to \infty} y_n = 1$, plugging into (1):

$\Rightarrow a_k + a_{k-1} + \dots + a_0 = 0$

$\Leftrightarrow \boxed{p(1) = 0}$

Now consider the problem

$$(P3) \begin{cases} y' = 1 \\ y(0) = 0 \end{cases} \quad (\text{sol is } y(t) = t)$$

we get a new eq:

$$a_k y_n + a_{k-1} y_{n-1} + \dots + a_0 y_{n-k} = h [b_k + b_{k-1} + \dots + b_0] \quad (2)$$

Convergent $\Rightarrow$ stable $\Rightarrow$ $p(1) = 0$ (1 is a <u>simple</u> root)

$p'(1) \neq 0$

A solution to (2) is given by:

$$y_n = (n+k)h\gamma, \quad \text{where } \gamma = \frac{q(1)}{p'(1)}.$$

Checking by substitution in LHS of (2):

$$h\gamma(a_k(n+k) + a_{k-1}(n+k-1) + \cdots + a_0 n)$$

$$= n h\gamma(\underbrace{a_k + a_{k-1} + \cdots + a_0}_{= p(1) = 0}) + h\gamma[\underbrace{k a_k + (k-1)a_{k-1} + \cdots + a_1}_{= p'(1)}]$$

$$= h\, q(1) = h\,[b_k + b_{k-1} + \cdots + b_0].$$

Now the first few steps are consistent with initial value: $y(0) = 0$:

$$|y(h, nh)| = (n+k)h\gamma \to 0 \text{ as } h \to 0.$$

(since $0 \leq n \leq k-1$)

Since the method is <u>convergent</u> we must have:

$$\lim_{n\to\infty} y_n = t \quad \text{when } nh = t$$

$$\Leftrightarrow \lim_{n\to\infty}(n+k)h\gamma \underset{\uparrow}{=} \lim_{n\to\infty} \overset{t}{\widetilde{nh}}\gamma = t \quad \Rightarrow \gamma = 1$$

$$\Leftrightarrow \boxed{p'(1) = q(1)}$$

since $\lim_{n\to 0} kh\gamma = 0$

<u>Example</u>: Milne's method $\quad y_n - y_{n-2} = h\left[\frac{1}{3}f_n + \frac{4}{3}f_{n-1} + \frac{1}{3}f_{n-2}\right]$

$p(z) = z^2 - 1$ roots: $+1, -1$ (simple) $\Rightarrow$ <u>stable</u>

$q(z) = \frac{1}{3}z^2 + \frac{4}{3}z + \frac{1}{3}$

$p'(z) = 2z$

$$\left.\begin{array}{l} q(1) = 2 = p'(1) \\ p(1) = 0 \end{array}\right\} \text{ \underline{consistent}}$$

$\Rightarrow$ <u>method is convergent</u>

Difference equation fundamentals (optional)

$x = (x_1, x_2, x_3, \dots)$ are sequences
$y = (y_1, y_2, y_3, \dots)$

A difference eq can be written using the shift operator

$$E x = (x_2, x_3, x_4, \dots) \text{, where } x = (x_1, x_2, x_3, \dots)$$

It is not hard to see that:

$$(E x)_n = x_{n+1}$$
$$(E^k x)_n = x_{n+k}$$
$$E^0 x = x$$

A linear difference operator is:

$$L = \sum_{i=1}^{m} a_i E^i$$

A difference eq is of the form:

$$L x = 0.$$

example: $x_{n+2} - 3x_{n+1} + 2x_n = 0$

$\Leftrightarrow (E^2 - 3E^1 + 2E^0) x = 0$

$\Leftrightarrow p(E) x = 0$ where $p(\lambda) = \lambda^2 - 3\lambda + 2$

Theorem (Simple roots) If p is a poly and λ a root then a sol to $p(E)x = 0$ is $(\lambda, \lambda^2, \lambda^3, \dots)$. If all roots of p are simple, $\neq 0$ then all solutions to $p(E)x = 0$ are in the span of all such solutions.

Theorem (Multiple roots) Let p be a poly with $p(0) \neq 0$. Then a basis for nullspace of $p(E)$ is:
with each root λ of p with multiplicity k, associate k solutions:
$x(\lambda), x'(\lambda), x''(\lambda), \dots, x^{(k-1)}(\lambda)$, where $x(\lambda) = (\lambda, \lambda^2, \lambda^3, \dots)$

$$x'(\lambda) = (1, 2\lambda, 3\lambda^2, \dots)$$
$$x''(\lambda) = (0, 2, 6\lambda, \dots)$$
$$\vdots$$