Math 5620 Numerical Analysis II

Chap 5 Initial value problems for Ordinary Diff Eq (ODE)

Objective develop methods to solve numerically problems of the kind

$$(1)\quad \begin{cases} \dfrac{dy}{dt} = f(t,y), & \text{for } t\in[a,b] \\ y(a) = \alpha \end{cases}$$

often an exact solution to (1) is too complicated or even cannot be found.

Extension to systems:

$$\begin{cases} \dfrac{dy_1}{dt} = f_1(t, y_1, y_2, \dots, y_n) \\ \dfrac{dy_2}{dt} = f_2(t, y_1, y_2, \dots, y_n) \\ \vdots \\ \dfrac{dy_n}{dt} = f_n(t, y_1, y_2, \dots, y_n) \end{cases}$$

for $t \in [a,b]$, s.t. $y_i(a) = \alpha_i$, $i = 1..n$

$$\left(\frac{d\underline{y}}{dt} = \underline{f}(t, \underline{y})\right)$$

And to n-th order IVP:

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}) \quad \text{s.t.} \quad \begin{cases} y(a) = \alpha_1 \\ y'(a) = \alpha_2 \\ \vdots \\ y^{(n-1)}(a) = \alpha_n \end{cases}$$

Fundamental questions to answer about the IVP (1)

- existence: does (1) admit a sol?
- uniqueness: is the sol to (1) unique?
- "stability": Do small changes in the statement of the problem introduce small changes in the solution?

⚠ this is different from notion of stability for numerical methods

A problem satisfying all 3 properties is said to be <u>well posed</u> (in the sense of Hadamard)

The IVP (1) is well posed under mild assumptions on f. We need some terminology (maybe you've seen this many times before!)

Def (Lipschitz condition)

A function $f(t,y)$ satisfies a Lipschitz condition in the variable y on some $D \subset \mathbb{R}^2$ iff $\exists L>0$

$$|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2| \quad \forall \ (t,y_1) \in D, \ (t,y_2) \in D.$$

L = Lipschitz constant for f.

(means $f(t,\bullet)$ is Lipschitz continuous for all t).

Example:

$$D = \{(t,y) \mid 0 \le t \le 1, -1 \le y \le 1\}$$

$$f(t,y) = t|y|$$

$$|f(t,y_1) - f(t,y_2)| = |t|y_1| - t|y_2|| = |t|\,||y_1| - |y_2|| \le |y_1 - y_2|$$

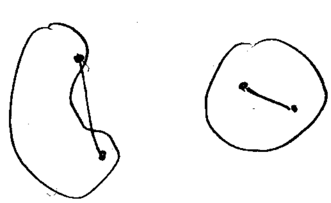
$L = 1$

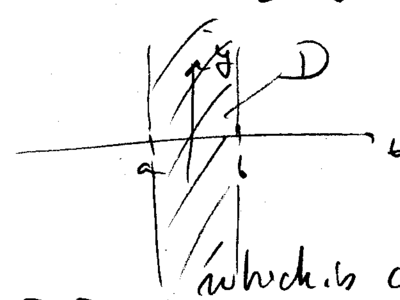
Def (Convex set) A set D is a convex set iff:

$$x, y \in D \Rightarrow \lambda x + (1-\lambda) y \in D \quad \forall \lambda \in [0,1]$$

in our case:

$$(t_1,y_1) \in D, (t_2,y_2) \in D \rightarrow (\lambda t_1 + (1-\lambda)t_2, \lambda y_1 + (1-\lambda)y_2) \quad \forall \lambda \in [0,1]$$





For IVP we usually have:

$$D = \{(t,y) \mid a < t < b, y \in \mathbb{R}\}$$

which is convex

Here is a sufficient (but not necessary) condition for Lipschitz cdt°:

<u>Theorem</u>: Let $f(t,y)$ be defined on some convex set $D \subset \mathbb{R}^2$.
If $\exists\, L > 0$ s.t.

$$\left|\frac{\partial f}{\partial y}(t,y)\right| \leq L \quad \forall (t,y) \in D$$

then f satisfies a Lipschitz condition on D in y with Lip. const. L.

<u>note</u>: $f(t,y) = t|y|$ satisfies the Lipschitz condition and yet $\frac{\partial f}{\partial y}$ does not exist at $y=0$.

<u>Existence and uniqueness</u> for IVP (1) are taken care by:

<u>Theorem</u> Let $D = \{(t,y) \mid a \leq t \leq b \ \&\ y \in \mathbb{R}\}$ and that

- $f(t,y)$ is continuous on D
- f satisfies Lip. cdt° on D in variable y, then the IVP(1)

<u>admits a unique solution</u>.

"<u>Stability</u>": can be formulated as follows:

let $y(t)$ solve $\begin{cases} \frac{dy}{dt} = f(t,y), & a \leq t \leq b \\ y(a) = \alpha \end{cases}$

$\exists\, \varepsilon_0 > 0, k > 0 \quad \forall \varepsilon$ s.t. $\varepsilon_0 > \varepsilon > 0$

$\forall\, \delta(t)$ continuous s.t. $|\delta(t)| \leq \varepsilon$

$\forall\, \delta_0 \in \mathbb{R}$ s.t. $|\delta_0| \leq \varepsilon$

$$\begin{cases} \frac{dz}{dt} = f(t,z) + \delta(t), & a \leq t \leq b, \\ z(a) = \alpha + \delta_0 \end{cases} \qquad \text{(perturbed problem)}$$

has a unique solution $z(t)$ and

$$|z(t) - y(t)| < k\varepsilon$$

(Essentially the mapping data $(\alpha, f(t,y))$ to solution y is continuous).

This "stability" property is crucial to trust solutions given by a numerical method (sources of error can be from the method itself or from numerical roundoff).

$\rightsquigarrow$ All numerical methods assume IVP is well-posed.

Example: $D = \{(t,y) \mid t \in [0,2], y \in \mathbb{R}\}$

$$\begin{cases} \dfrac{dy}{dt} = \underbrace{y - t^2 + 1}_{= f(t,y)}, \quad 0 \le t \le 2 \qquad (IVP) \\ y(0) = \frac{1}{2} \end{cases}$$

$\left|\dfrac{\partial f}{\partial y}\right| = |1| = 1 \Rightarrow f(t,y)$ satisfies Lip. cdto on D with variable y.

& f constant $\Rightarrow$ problem is stable to perturbations in init. data.

We can verify this directly:

$$\begin{cases} \dfrac{dz}{dt} = z - t^2 + 1 + \delta \qquad \text{(perturbed problem)} \\ z(0) = \frac{1}{2} + \delta_0 \end{cases}$$

IVP has sol: $y(t) = \frac{-1}{2} e^{t} + (t+1)^2$

$z(t) = \left(\frac{-1}{2} + \delta_0\right)e^{t} + (t+1)^2 + (e^{t} - 1)\delta$

$$|z - y| = |(\delta_0 + \delta)e^{t} - \delta| \le |\delta_0 + \delta| e^2 + |\delta| \le 2e^2\varepsilon + \varepsilon = (2e^2 + 1)\varepsilon$$

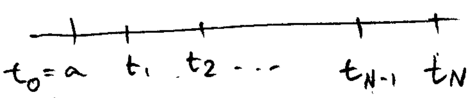
§5.2 Euler's method

A simple numerical method to solve IVP:

$$\begin{cases} \dfrac{dy}{dt} = f(t,y), & a \le t \le b \\ y(a) = \alpha \end{cases} \qquad \text{(assuming well-posedness)}$$

Euler's method gives approx only on some mesh points:

$$t_j = a + hj \quad \text{where } h = \frac{b-a}{N} = \text{step size}$$



$(N+1)$ $\underbrace{\text{equally spaced}}_{\text{(can be relaxed)}}$ points)

Idea for Euler's method: Taylor's theorem:

$$y(t_{i+1}) = y(t_i) + y'(t_i)\underbrace{(t_{i+1}-t_i)}_{=h} + \frac{1}{2}y''(\xi_i)\underbrace{(t_{i+1}-t_i)^2}_{=h^2}$$

for some $\xi_i \in (t_i, t_{i+1})$.

$$= y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i)$$

$$= y(t_i) + hf(t_i, y(t_i)) \underbrace{+ \tfrac{1}{2}h^2 y''(\xi_i)}_{\text{neglect for Euler's method}} \qquad \text{(y satisfies DE in IVP)}$$

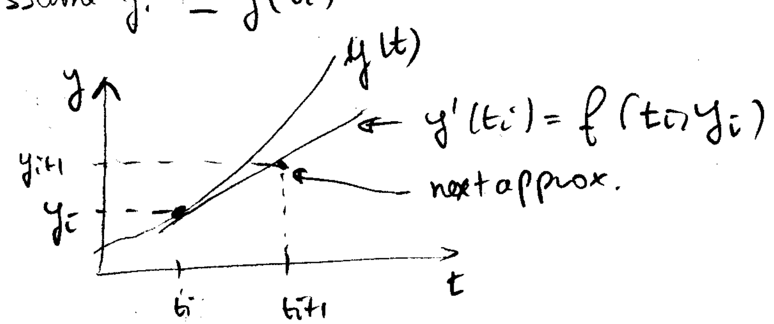
$y_0 = \alpha$

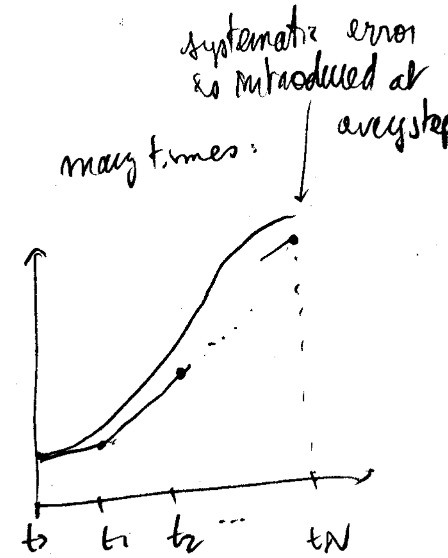
for $i = 0 .. N-1$

$$y_{i+1} = y_i + h\underset{\text{known}}{f(t_i, y_i)}$$

Geometrical interpretation

Assume $y_i = y(t_i)$





Several types of errors in numerical methods for DE:

- **Local truncation error**: error made in one step.
  For example in Euler's method: $O(h^2)$ since:
  $$y(t_i+h) = y(t_i) + h f(t, y(t_i)) + \underline{O(h^2)}$$
  (⚠ book defines this as $\frac{1}{h}$)

- **Local roundoff error**: precision used for computation (10^{-16} double, 10^{-8} single)

- **Global truncation error**: accumulation of all the local truncation errors. If local truncation error was $O(h^{n+1})$ then the global truncation error must be $O(h^n)$ because the number of steps $O(h)$.

- **Global roundoff error**: accumulation of local roundoff errors of previous steps.

- **Total error** = global truncation error + global roundoff error

If the global truncation error is $\underline{O(h^n)}$ then the method is of **order n**

Example: Euler's method is of order 1, and it is relatively simple to derive more precise bounds on the global truncation error:

Thm let f be continuous and satisfy Lipschitz condition with L-constant L on $D = \{(t, y) \mid t \in [a, b],\ y \in \mathbb{R}\}$ and that
$$|y''(t)| \leq M \quad \forall t \in [a, b] \text{ for some } M > 0.$$

Let $y(t)$ be the sol to IVP
$$\begin{cases} \frac{dy}{dt} = f(t, y) \\ y(a) = \alpha \end{cases}$$

and $y_0, y_1, \ldots, y_N$ the approximations given by Euler's method then:

$$|y(t_i) - y_i| \leq \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right], \quad i = 0, 1, \dots, N.$$

<u>proof</u> (sketch)

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$

$$y_{i+1} = y_i + hf(t_i, y_i)$$

$$\Rightarrow |y(t_{i+1}) - y_{i+1}| \leq |y(t_i) - y_i| + h\left|f(t_i, y(t_i)) - f(t_i, y_i)\right| + \frac{h^2}{2}|y''(\xi_i)|$$

$$\leq (1 + hL)|y(t_i) - y_i| + \frac{h^2}{2} M$$

using Lemma 5.8 in book: ($\sim$ geometric series + bounding exp)

$$|y_{i+1} - w_{i+1}| \leq e^{(i+1)hL}\left(\underbrace{|y_0 - w_0|}_{=0} + \frac{h^2 M}{2hL}\right) - \frac{h^2 M}{2hL}$$

$$\leq \frac{h^2 M}{2hL}\left(e^{(i+1)hL} - 1\right)$$

$(i+1)hL$: $t_{i+1} - t_0 = t_{i+1} - a$.

<u>Lemma 5.8</u> $\{a_i\}_{i=0}^{k}$ is a seq with $a_0 \geq -\frac{t}{s}$ and

$$a_{i+1} \leq a_i(1+s) + t \quad \text{for } i = 0, \dots, k-1$$

$$\Rightarrow a_{i+1} \leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

<u>proof</u>:

$$a_{i+1} \leq a_i(1+s) + t \leq ((1+s)a_{i-1} + t)(1+s) + t$$

$$\leq \left((1+s)((1+s)a_{i-2} + t) + t\right)(1+s) + t$$

$$\vdots$$

$$\leq (1+s)^{i+1} a_0 + t\sum_{j=0}^{i} (1+s)^j$$

$$\Rightarrow a_{i+1} \leq (1+s)^{i+1} a_0 + t \left[\frac{1-(1+s)^{i+1}}{1-(1+s)}\right]$$

$$= (1+s)^{i+1}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

$$\leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}$$

since $(1+x)^{\alpha} = e^{\alpha \ln(1+x)} \leq e^{\alpha x}$

can be shown using Taylor's thm.

For Euler's method it's even possible to incorporate the round-off errors in the analysis:

instead of having:

$y_0 = \alpha$

for $i = 0..N-1$,

$\quad y_{i+1} = y_i + h f(t_i, y_i)$

we commit a mistake at each step:

$y_0 = \alpha + \delta_0$

for $i = 0..N-1$

$\quad y_{i+1} = y_i + h f(t_i, y_i) + \delta_{i+1}$

If $|\delta_i| < \delta$ it is possible to show:

$$|y(t_i) - y_i| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{L(t_i-a)} - 1\right] + |\delta_0| e^{L(t_i-a)}$$

performance degrades as $h \to 0$!!
(similar to problems with numerical differentiation)

The h giving smallest error is

$$h = \sqrt{\frac{2\delta}{M}} \quad \text{(simple calculation)}$$

§3 Higher order Taylor methods :

Same derivation as Euler's method but carry Taylor series further :

$$y(t_{i+1}) = y(t_i) + h\, y'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$$

for some $\xi_i \in (t_i, t_{i+1})$.

$$y'(t) = f(t, y(t))$$

$$y''(t) = \frac{d}{dt}\left[f(t, y(t))\right]$$

$$\vdots$$

$$y^{(k)}(t) = \frac{d^{k-1}}{dt^{k-1}}\left[f(t, y(t))\right]$$

Taylor method of order n

$y_0 = \alpha$

for $i = 0 \ldots N-1$,

$$\quad y_{i+1} = y_i + h\, f(t, y(t)) + \frac{h^2}{2} \frac{d}{dt}\left[f(t, y(t))\right] + \cdots + \frac{h^n}{n!} \frac{d^{n-1}}{dt^{n-1}}\left[f(t, y)\right]$$

Local truncation error is $O(h^{n+1})$

In general evaluating $\frac{d^k}{dt^k}\left[f(t, y(t))\right]$ can be quite involved because of the repeated application of the chain rule.

For example:

$$\begin{cases} y' = \cos t - \sin y + t^2 = f(t, y) \\ y(-1) = 3 \end{cases}$$

$$y'' = \frac{d}{dt}\left[f(t, y(t))\right] = -\sin t - y' \cos y + 2t$$

$$y''' = \frac{d^2}{dt^2}\left[f(t, y(t))\right] = -\cos t - y'' \cos y + (y')^2 \sin y + 2$$

$$y^{(4)} = \frac{d^3}{dt^3}\left[f(t, y(t))\right] = \sin t - y^{(3)} \cos y + 3y' y'' \sin y + (y')^3 \cos y$$

etc...

Note: • Assumes f is smooth ($n-1$ times differentiable if method of order n is desired)

• The higher the accuracy the more complicated the steps become
• differentiation can be carried out symbolically if possible!

§5.4 Runge-Kutta methods

Taylor series method for $\begin{cases} \frac{dy}{dt} = f(t,y), & t \in [a,b] \\ y(a) = A \end{cases}$

require us to compute formulas for
$$y'' = \frac{d}{dt}[f(t,y)]$$
$$y^{(3)} = \frac{d^2}{dt^2}[f(t,y)]$$
etc...

which can be quite involved.

Runge-Kutta methods avoid this difficulty by carefully chosen combinations of values of $f(t,y)$.

Second order Runge Kutta methods:

Start with Taylor series:
$$y(t+h) = y(t) + h\,y'(t) + \frac{h^2}{2} y''(t) + \frac{h^3}{3!} y^{(3)}(t) + \dots$$

From the DE we get:
$$y'(t) = f(t,y) = \boxed{f}$$
$$y''(t) = \frac{d}{dt} f(t,y) = \frac{\partial f}{\partial t}(t,y) + y' \frac{\partial f}{\partial y}(t,y) = \boxed{f_t + f f_y}$$
$$y'''(t) = \frac{d^2}{dt^2} f(t,y) = f_{tt} + f f_{ty} + (f_t + f f_y) f_y + f\,(f_{ty} + f f_{yy})$$
etc...

$$\Rightarrow\ y(t+h) = y(t) + hf + \frac{h^2}{2}(f_t + f f_y) + O(h^3)$$

The idea is to eliminate the partial deriv of f by using the first few terms of the two variable Taylor series for $f(t,y)$:

$$f(t+h, y+hf) = f + \nabla f \cdot \begin{pmatrix} h \\ hf \end{pmatrix} + O(h^2)$$
$$= f + hf_t + hf f_y + O(h^2)$$

Rewriting the Taylor series of y:

$$(*) \quad y(t+h) = y(t) + \frac{1}{2}hf + \frac{1}{2}h[\underbrace{f + hf_t + hff_y}_{\text{first 2 terms of 2 variable Taylor series of } f}] + O(h^3)$$

$$= y(t) + \frac{1}{2}hf + \frac{1}{2}h\left[f(t+h, y+hf) + O(h^2)\right] + O(h^3)$$

Thus it's possible to construct an update which has the same $O(h^3)$ local truncation error as 2nd-order Taylor method:

<u>Modified Euler method</u>

$y_0 = A$

for $i = 0, 1, \ldots, N-1$

$F_1 = hf(t_i, y_i)$

$F_2 = hf(t_i+h, y_i + F_1)$

$y_{i+1} = y_i + \frac{1}{2}F_1 + \frac{1}{2}F_2$

The general form for Second order Runge Kutta updates is:

$$y(t+h) = y + w_1 hf + w_2 hf(t+\alpha h, y+\beta hf)$$

where w_1, w_2, α, β are parameters that we can adjust.

Using two variable Taylor expansion:

$$y(t+h) = y + w_1 hf + w_2 h\left[f + \alpha h f_t + \beta h f f_y\right]$$

Matching comparable terms with $(*)$ we get:

$$\begin{cases} w_1 + w_2 = 1 \\ w_2 \alpha = \frac{1}{2} \\ w_2 \beta = \frac{1}{2} \end{cases}$$

$\rightsquigarrow$ so there is a family of RK order 2 methods

<u>Modified Euler</u>: $w_1 = w_2 = \frac{1}{2}$, $\alpha = \beta = 1$

<u>Midpoint method</u>: $w_1 = 0, w_2 = 1$, $\alpha = \beta = \frac{1}{2}$

(also $O(h^3)$ LTE)

$y_0 = A$

for $i = 0, 1, \dots, N-1$

$F_1 = h f(t_i, y_i)$

$F_2 = h f(t_i + \frac{h}{2}, y_i + \frac{1}{2} F_1)$

$y_{i+1} = y_i + F_2$

<u>Heun's method</u>: $w_1 = \frac{1}{4}$, $w_2 = \frac{3}{4}$, $\alpha = \beta = \frac{2}{3}$

$y_0 = A$

for $i = 0, 1, \dots, N-1$

$F_1 = h f(t_i, y_i)$

$F_2 = h f(t_i + \frac{2}{3} h, y_i + \frac{2}{3} F_1)$

$y_{i+1} = y_i + \frac{1}{4} F_1 + \frac{3}{4} F_2$

Runge Kutta methods of order 3 are obtained by matching terms between Taylor expansion of order 3 of $y(t)$ and $f(t+\alpha h, y + w_1 f(t+\beta h, y + w_2 f(t,y)))$

The derivation is quite tedious, but here is one possible RK order 3 method:

$y_0 = A$

for $i = 0, \dots, N-1$

$$\begin{pmatrix} \alpha = w_1 = \frac{3}{4} \\ \beta = w_2 = \frac{1}{2} \end{pmatrix}$$

$$F_1 = h f(t_i, y_i)$$

$$F_2 = h f(t_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}F_1)$$

$$F_3 = h f(t_i + \tfrac{3}{4}h, y_i + \tfrac{3}{4}F_2)$$

$$y_{i+1} = y_i + \tfrac{1}{9}(2F_1 + 3F_2 + 4F_3)$$

However it is not commonly used in practice.

The most popular RK method is that of order 4; again the derivation is tedious but the implementation straight forward.:

Runge Kutta method of order 4 $\quad$ (LTE $O(h^5)$)

$y_0 = A$

for $i = 0, \dots, N-1$

$$F_1 = h f(t_i, y_i)$$

$$F_2 = h f(t_i + \tfrac{1}{2}h, x + \tfrac{1}{2}F_1)$$

$$F_3 = h f(t_i + \tfrac{1}{2}h, x + \tfrac{1}{2}F_2)$$

$$F_4 = h f(t_i + h, x + F_3)$$

$$y_{i+1} = y_i + \tfrac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

# of function eval.	1	2	3	4	5	6	7	8	9	10	11	12	....
max order of RK method	1	2	3	4	4	5	6	6	7	7	8	9	...

So # of function eval increases more rapidly than the max order of RK method. $\Rightarrow$ higher order RK methods are less attractive than the classical RK4. (it makes more sense to use smaller time steps for RK4 than using a higher order method with bigger steps).

§5.5 Adaptive Runge Kutta Fehlberg method

Idea: Estimate local truncation error and adjust the step length accordingly.

Local truncation error estimation

Assume we are given two update formulas for solving

$$\begin{cases} y' = f(t,y) & , t \in [a,b] \\ y(a) = \alpha \end{cases}$$

with local truncation errors differing by 1:

① $y(t_{i+1}) = y(t_i) + h\,\phi(t_i, y(t_i), h) + O(h^{n+1})$ (order n)

$y_0 = \alpha$

for $i = 0, \dots, N-1$

| $y_{i+1} = y_i + h\,\phi(t_i, y_i, h)$

② $y(t_{i+1}) = y(t_i) + h\,\tilde{\phi}(t_i, y(t_i), h) + O(h^{n+2})$ (order $n+1$)

$\tilde{y}_0 = \alpha$

for $i = 0 .. N-1$

| $\tilde{y}_{i+1} = \tilde{y}_i + h\,\tilde{\phi}(t_i, y_i, h)$

these update formulas could come from Taylor's method, RK etc...

Assuming $y_i = \tilde{y}_i = y(t_i)$:

$$\underbrace{y_{i+1} - y(t_{i+1})}_{\substack{\text{Local trunc err for (1)} \\ \parallel \\ \mathcal{O}(h^{n+1})}} = \underbrace{\tilde{y}_{i+1} - y(t_{i+1})}_{\substack{\text{local trunc error for (2)} \\ \parallel \\ \mathcal{O}(h^{n+2})}} + \underbrace{y_{i+1} - \tilde{y}_{i+1}}_{\substack{\parallel \\ \Rightarrow \mathcal{O}(h^{n+1})}}$$

Thus:

$$\underbrace{y_{i+1} - y(t_{i+1})}_{\text{Local trunc err for (1)}} = y_{i+1} - \tilde{y}_{i+1} + \mathcal{O}(h^{n+2})$$

How to use this estimate to predict step size needed to get a local truncation error $< \delta$ (where $\delta > 0$ is prescribed) ?

Simplifying assumption: K constant indep of h.

$$\tau_{i+1}(h) = y_{i+1} - y(t_{i+1}) \approx K h^{n+1} \qquad \text{((1) is } n \text{ order method)}$$

If instead of h the step were qh $(q > 0)$:

$$\tau_{i+1}(qh) \approx K(qh)^{n+1} = q^{n+1} K h^{n+1} \approx q^{n+1} \tau_{i+1}(h) \approx q^{n+1}(y_{i+1} - \tilde{y}_{i+1})$$

$$\Rightarrow \quad q^{n+1} |y_{i+1} - \tilde{y}_{i+1}| \approx |\tau_{i+1}(qh)| \leq \delta$$

$\Rightarrow$ for $q \leq \left(\dfrac{\delta}{|y_{i+1} - \tilde{y}_{i+1}|} \right)^{1/(n+1)}$ we can expect loc. trunc. err to be less than δ.

Thus if we want to keep the local trunc error $\leq \delta$, we can set

$$h_{new} = h \left(\frac{\delta}{|y_{i+1} - \tilde{y}_{i+1}|} \right)^{\frac{1}{n+1}}$$

However it is recommended (based on empirical observations) to relax this to:

$$h_{new} = h\left(\frac{\delta}{2|y_{i+1}-\tilde{y}_{i+1}|}\right)^{\frac{1}{m}} \simeq 0.9\, h\left(\frac{\delta}{|y_{i+1}-\tilde{y}_{i+1}|}\right)^{\frac{1}{m}}$$

Fehlberg (1969) used an order 4 RK method with 5 fun evals
a — 5 — 6 —

total 11 fun. evals / iteration?? Does not seem very practical!
However the coefficients in Fehlberg's method are carefully chosen so that <u>only</u> 6 fun evals are needed / iter.
(by making the eval points for the order 4 method match those of the order 5 method).

The updates are of the form:

$$y_{i+1} = y_i + \sum_{j=1}^{6} a_j F_j$$
$$\tilde{y}_{i+1} = \tilde{y}_i + \sum_{j=1}^{6} b_j F_j$$

where the F_i are of the form:

$$F_i = h\, f\left(t + c_i h,\ x + \sum_{j=1}^{i-1} d_{ij} F_j\right)$$

(see book for the coeff. Sometimes only $a_i - b_i$ is specified)

The complete algo is:

$y_0 = \alpha$, $h = h_{max}$, $flag = 1$

while (flag = 1)

- compute $F_1, \dots, F_6$
- compute $e = |y_{i+1} - \bar{y}_{i+1}|$ (a linear combination of the F_i, w/ coeff given in tables)
- if $e \leq \delta$
  - accept step:
    - $t = t + h$
    - $y_{i+1} = y_i + \sum_{j=1}^{6} b_j F_j$ (use higher order method)
- Set $q = 0.9 \, (\delta / e)^{\frac{1}{4}}$
- Set $h = \begin{cases} 0.1h & \text{if } q \leq 0.1 \\ 4h & \text{if } q \geq 4 \\ qh & \text{otherwise} \end{cases}$
- Set $h = \min(h, h_{max})$
- if $t \geq b$
  - $flag = 0$

  else
  - if $t + h > b$
    - $h = b - t$ (adjust last step)
- if $h < h_{min}$
  - $flag = 0$; (Step size is too small abnormal termination

(normal termination)