MATH 5620 NUMERICAL ANALYSIS II
HOMEWORK 5, DUE FRIDAY APRIL 3 2009

The goal of this assignment is to use the finite element method with piecewise linear finite elements to solve the one dimensional BVP

$$\begin{cases} -u'' = f, \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

The weak formulation of the problem is

$$\text{Find } u \in V \text{ s.t. } a(u, v) = (f, v), \forall v \in V, \tag{2}$$

where $V = \{v \mid ||v||_{L^2}^2 + ||v'||_{L^2}^2 < \infty \text{ and } v(0) = v(1) = 0\}$.

Let $x_i = (i-1)h$, where $h = 1/n$ and $i = 1, \ldots, n+1$. We subdivide the interval $[0, 1]$ into n elements $I_e = [x_e, x_{e+1}]$. The approximation space for the Galerkin method is $V_h = \{v \in V \mid v|_{I_e} \in P_1\}$, where P_1 is the space of polynomials with degree less than 1. The Galerkin problem is

$$\text{Find } u_h \in V_h \text{ s.t. } a(u_h, v_h) = (f, v_h), \forall v_h \in V_h. \tag{3}$$

As we saw in class, the local to global mapping can be represented as a matrix `i = [1:n;2:n+1]';`, where $i(e, j)$ is the global index of the $j-$th local degree of freedom of element e.

1. Assemble the stiffness matrix $K \in \mathbb{R}^{n+1 \times n+1}$, where $K_{i,j} = a(\phi_i, \phi_j)$ and the ϕ_i are the hat functions ($\phi_i(x_j) = \delta_{ij}$). Do this element by element and store it as a sparse matrix as follows

```
K = sparse(n+1,n+1);
for e=1:n,
 K(i(e,:),i(e,:)) = K(i(e,:),i(e,:)) + Kloc/(x(e+1)-x(e));
end;
```

   where `Kloc=[1 -1;-1 1];` is the local stiffness matrix.
2. Assume the right hand side $f \in V_h$, i.e. it is determined by its values at the nodes x_i. Write a procedure to assemble *element by element* the right hand side $F \in \mathbb{R}^{n+1}$, where $F_j = (f, \phi_j)$. Just as for the stiffness matrix this should be done as a loop over the elements where the contribution from the current element is added to the vector F. These integrals are more conveniently evaluated by changing variables from the current element $[x_e, x_{e+1}]$ to the "parent" or reference element $[0, 1]$. There the integrals can be evaluated exactly by hand.
3. The system $Ku = F$ you will obtain is singular because we have not taken into account the Dirichlet boundary conditions. An easy way of imposing that $u(0) = u(1) = 0$ is to hard wire the condition in the equations of the linear system:

```
K(1,:) = 0; K(1,1)=1;
K(n+1,:) = 0; K(n+1,n+1)=1;
F(1)=0; F(n+1)=0;
```

4. Solve the problem (1) with $n = 10$ and $n = 100$ and $f(x) = \exp(x)$ (taking as right hand side for the system $F_j = \exp(x_j)$). Plot the numerical solution along with the true solution $u_{true}(x) = -\exp(x) + (\exp(1) - 1)x + 1$.
5. Do a log-log plot of h and the maximum error between the numerical solution u_h and the true solution u_{true}, for $n = 10, 100, 500, 1000$. How does the error behave as a function of h?