Problem 1
\[ y_n - y_{n-2} = h (f_n - 3f_{n-1} + 4f_{n-2}) \]
(a) This is an implicit method since \( y_n \) is defined implicitly above.
(b) The polynomials associated with this method are:
\[ p(z) = z^2 - 1 \]
\[ q(z) = z^2 - 3z + 4 \]

Stability:
The roots of \( p \) are \( \pm 1 \), both of magnitude 1, but \( q \) is not.
\( \Rightarrow \) method is stable.

Consistency:
\[ p(1) = 0 \]
\[ p'(1) = 2 \]
\[ q(1) = 1 - 3 + 4 = 2 \]
Method is consistent.

Since method is stable and consistent, it must be convergent.

Problem 2
A is assumed to be symmetric \( (A = A^T) \)

(a) Power method
\[ \mathbf{v}^{(0)} \text{ some vector with } \| \mathbf{v}^{(0)} \| = 1 \]
\[ \mathbf{v}^{(k)} = A \mathbf{v}^{(k-1)} \]
\[ \mathbf{v}^{(k)} = \frac{\mathbf{v}^{(k)}}{\| \mathbf{v}^{(k)} \|} \]
\[ \lambda^{(k)} = \mathbf{v}^{(k)^T} A \mathbf{v}^{(k)} \]

QR algorithm
\[ A^{(1)} = Q_0 \cdot A Q_0 = (\| \) \]

(reduction to tridiagonal form)

\[ Q^{(k)} R^{(k)} = A^{(k-1)} \]
\[ A^{(k)} = R^{(k)} Q^{(k)} \]

Eigenvalues are in diag \( (A^{(k)}) \)
Fundamental differences between methods:

- Power method converges to largest eigenvalue (and eigenvector) in magnitude
- QR gives all eigenvalues of the matrix

**Power method**
- **Advantages**:
  - Does not need matrix \( A \), only how to apply \( A \) to a vector
  - Cheap for large matrices
  - Quadratic convergence to eigenvalue
- **Disadvantages**:
  - Only gives largest eigenvalue
  - Can have slow convergence if largest eigenvalue magnitude is not simple

**QR algorithm**
- **Advantages**:
  - Gives all eigenvalues at once
- **Disadvantages**:
  - Can be expensive, especially reduction to tridiagonal form
  - Needs shifts to accelerate convergence
  - Needs to store whole matrix

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**Problem 3**

(a) Taylor expanding \( y(t_{i+1}) \), \( y(t_{i-1}) \) around \( y(t_i) \) are get

\[
\begin{align*}
y(t_{i+1}) &= y(t_i) + R y'(t_i) + \frac{R^2}{2} y''(t_i) + \frac{R^3}{6} y'''(t_i) + O(R^4) \\
y(t_{i-1}) &= y(t_i) - R y'(t_i) + \frac{R^2}{2} y''(t_i) - \frac{R^3}{6} y'''(t_i) + O(R^4)
\end{align*}
\]

\( + \)

\[y(t_{i+1}) + y(t_{i-1}) = 2y(t_i) + R^2 y''(t_i) + O(R^4)\]

\[\implies y''(t_i) = \frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{R^2} + O(R^2)\]
(b) The nodes \( t_i \) and \( t_n \) are special cases because of the boundary conditions.

\[
\begin{align*}
\frac{1}{\alpha} \left( \frac{y_{i+1} - 2y_i + y_i}{\alpha} \right) &= P_i \frac{1}{2h} \left( \frac{y_{i+1} - y_i}{\alpha} \right) + q_i y_i + r_i \\
\frac{1}{\alpha} \left( y_{i+1} - 2y_i + y_{i-1} \right) &= P_i \frac{1}{2h} \left( y_{i+1} - y_{i-1} \right) + q_i y_i + r_i \quad \text{for } i = 2, \ldots, n-1 \\
\frac{1}{\alpha} \left( \frac{y_{n+1} - 2y_n + y_n}{\alpha} \right) &= P_n \frac{1}{2h} \left( \frac{y_{n+1} - y_n}{\alpha} \right) + q_n y_n + r_n
\end{align*}
\]

(c) In system form we get: \( AV = B \) with

\[ A = L - D - Q \]

where

\[
A = \frac{1}{\alpha^2} \begin{bmatrix}
-2 & 1 \\
1 & -2 & 1 \\
& & \ddots & \ddots \\
& & 1 & -2
\end{bmatrix}
\]

\[
D = \frac{1}{2h} \begin{bmatrix}
0 & P_1 & P_2 & \cdots & P_{n-1} \\
-P_2 & 0 & P_3 & \cdots & \vdots \\
& -P_3 & 0 & \ddots & \vdots \\
& & \ddots & \ddots & \vdots \\
& & & -P_{n-1} & 0
\end{bmatrix}
\]

\[
P_i = p(t_i)
\]

\[
Q = \begin{bmatrix}
q_1 \\
q_2 \\
\vdots \\
q_n
\end{bmatrix}
\]

\[
p_i = q(t_i)
\]
\[ B = R - \frac{\alpha p_1}{2h} e_1 + \frac{\beta P_m}{2h} e_n - \frac{\alpha}{k^2} e_1 - \frac{\beta}{k^2} e_n \]

\[ R = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}, \quad r_i = r(t_i) \]

or in other words:

\[ B = \begin{bmatrix} r_1 - \frac{\alpha p_1}{2h} - \frac{\alpha}{k^2} \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n + \frac{\beta P_m}{2h} - \frac{\beta}{k^2} \end{bmatrix} \]

Problem 4

(a) We simply multiply PDE by test function \( \sigma \in V \) and integrate:

\[ \int_0^1 u'' \sigma \, dx = \int_0^1 f \sigma \, dx \quad \forall \sigma \in V \]

\[ \text{II IBP} \]

\[-u' \sigma \bigg|_0^1 + \int_0^1 u' \sigma' \, dx \]

\[ = 0 \quad \text{because} \]

\[ \sigma \in V \]

Thus if \( u \) solves PDE:

\[ \alpha(u, v) = (f, v) \quad \forall v \in V, \quad \text{where} \]

\[ \alpha(u, v) = \int_0^1 u' v' \, dx \quad \text{and} \quad (f, v) = \int_0^1 f v \, dx. \]
(b) Finite element spaces are nodal values and the functions are approximate with piecewise linear. So the finite dimensional subspace \( V_h \subset V \) of functions we use is composed of "hat functions"

\[
V_h = \text{span} \{ \phi_i \}
\]

The Ritz-Galerkin problem:

Find \( u_h \in V_h \) s.t. \( a(u_h, v_h) = (f, v_h) \) \( \forall v_h \in V_h \).

produces the "best" approximation to the solution \( u \) in \( V_h \) (best in the sense of the energy norm) and is equivalent to solving the linear system:

\[
A U = F, \quad \text{where } A_{ij} = a(\phi_i, \phi_j) \\
F_i = (f, \phi_i)
\]

and \( u_h = \sum_{i=1}^{n} U_i \phi_i \).

Here is some code to solve the problem above.

/* Global to local map */

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{2}{3} \\
\vdots & \vdots \\
1 & 1
\end{bmatrix}
\]

/* Assemble stiffness matrix and RHS */

for \( e = 1 \ldots m+1 \)

\[
A(i(e,:), i(e,:)) = A(i(e,:), i(e,:)) + A_{e} \\
F(i(e,:)) = F(i(e,:)) + F_{e}
\]
Here \((A^2)_{ij} = \int_{I_e} (\phi_i^e)'(\phi_j^e)' \ dx\) = local stiffness matrix

\((F^e)_{i} = \int_{I_e} \phi_i^e \ f \ dx\) = local RHS

/* Take into account homog. Dirichlet B.C. */
\(A(1, :) = 0;\) \(A(1, 1) = 1;\) \(F(1) = 0;\)
\(A(m+1, :) = 0;\) \(A(m+1, m+1) = 1;\) \(F(m+1) = 0;\)

/* Solve System */
\[ U = \text{Linear Solve} \ (A, F); \]

**Problem 5**

(a) Method of lines simply means we discretize PDE first in space:

\[ U = x_0 \ x_1 \ x_2 \ldots \ x_m \ x_{m+1} = 1 \]

\[ U'(t) = -\frac{a}{2R} \bigg( U_2(t) - \overbrace{\ldots}^{U_{m+1}(t)} \bigg) \]

\[ U_i'(t) = -\frac{a}{2R} \bigg( U_{i+1}(t) - U_{i-1}(t) \bigg), \ i = 2, \ldots, m \]

\[ U'_{m+1}(t) = -\frac{a}{2R} \bigg( U_{m+2}(t) - U_m(t) \bigg) \]

\[ = U_1(t) \text{ by periodic B.C.} \]

**Be sure to use right discretization.**
Therefore method of lines gives:

\[ U'(t) = AU(t) \]

where

\[
A = -\frac{\alpha}{2h} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}
\]

and

\[ U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_{m+1}(t) \end{bmatrix} \]

(b) Discretization in time gives:

\[
\frac{U^{n+1} - U^{n-1}}{2k} = AU^n
\]

\[ U^{n+1} = U^{n-1} + 2k AU^n \]

(c) Absolute stability region

For stability of system of ODEs we need to find \( k \) s.t.

\[ \Re \lambda_p(A) \in \text{absolute stability region, } p = 1, \ldots, m+1 \]

Since

\[ |\Im \lambda_p(A)| \leq \frac{1}{R} \]

in order to have \( |\Im k \lambda_p(A)| \leq 1 \) we need

\[ \frac{k |\alpha|}{R} \leq 1 \Leftrightarrow \left| k \alpha \right| \leq R \]