MATH 5620 – NUMERICAL ANALYSIS II
PRACTICE FINAL EXAM

Problem 1. Consider the multistep method:

$$y_n - y_{n-2} = h(f_n - 3f_{n-1} + 4f_{n-2}).$$

(a) Is this method implicit or explicit?
(b) Is this method convergent, stable and/or consistent? Justify your answer.

Problem 2.
(a) Write pseudocode for the Power Method and the QR algorithms for symmetric matrices. Please do not worry about stopping criteria. You may assume the availability of a routine for reduction to tridiagonal form.
(b) Explain in your own words the differences, advantages and disadvantages of these two methods.

Problem 3. Consider the linear BVP

$$\begin{cases} y'' = py' + qy + r \\ y(0) = \alpha,\ y(1) = \beta \end{cases}$$

where p, q and r are smooth functions defined on $[a, b]$. Let $t_i = ih$, where $i = 0, \ldots, n+1$ and $h = 1/(n+1)$.
(a) Use the Taylor expansions of $y(t_{i+1})$ and $y(t_{i-1})$ around $t = t_i$ to show that

$$y''(t_i) = \frac{1}{h^2}(y(t_{i+1}) - 2y(t_i) + y(t_{i-1})) + \mathcal{O}(h^2).$$

(b) Recall the centered differences approximation

$$y'(t_i) = \frac{1}{2h}(y(t_{i+1}) - y(t_{i-1})) + \mathcal{O}(h^2).$$

Write the finite difference approximation to the problem, using the notation $y_i \approx y(t_i)$. Since the boundary conditions are $y_0 = \alpha$ and $y_{n+1} = \beta$, there are only n equations.
(c) Write the finite differences approximation as a system $AY = B$ with n unknowns.

Problem 4. Consider the BVP

$$\begin{cases} -u'' = f(x),\ 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases}$$

(a) Let $V = \{v \mid \|v\|_{L^2}^2 + \|v'\|_{L^2}^2 < \infty \text{ and } v(0) = v(1) = 0\}$. Derive the weak formulation of the problem, that is

$$\text{Find } u \in V \text{ such that } a(u, v) = (f, v) \text{ for all } v \in V,$$

explicitly writing $a(u, v)$ and (f, v).
(b) Write pseudocode for solving the problem with piecewise linear (P_1) elements.

Problem 5. Consider the following advection equation (a hyperbolic problem) with periodic boundary conditions,

$$\begin{cases} u_t + au_x = 0,\ t > 0,\ 0 < x < 1, \\ u(0, t) = u(1, t),\ t > 0, \\ u(x, 0) = \eta(x),\ 0 < x < 1. \end{cases}$$

(a) The interval $[0,1]$ is discretized with the points $x_i = ih$, $i = 0,\dots,m+1$, where $h = 1/(m+1) = \Delta x$. Let $U_i(t) \approx u(x_i,t)$. Use the method of lines to approximate the PDE by a system of ODEs $U'(t) = AU(t)$, where $U = [U_1, U_2, \dots, U_{m+1}]^T$ and A comes from the usual centered differences approximation to u_x.

(b) Discretize in time using the Midpoint method. For a one dimensional IVP $u' = f(u,t)$, the Midpoint method is

$$\frac{u_{n+1} - u_{n-1}}{2k} = f(u_n, t_n).$$

Please use the notation $U_i^n \approx u(x_i, t_n)$ (or $U^n = [U_1^n, U_2^n, \dots, U_m^n]^T$ in vector form), where $t_n = nk$ and $k = \Delta t$ is the time step.

(c) The absolute stability region for the Midpoint method is the segment between $-j$ and j in the complex plane (here $j = \sqrt{-1}$). Recall that eigenvalues of A are:

$$\lambda_p(A) = -\frac{aj}{h}\sin(2\pi ph),\ p = 1,\dots,m+1.$$

For a given h, find a condition on k for the system of ODEs to be stable.