MATH 5620 - Numerical analysis
Practice midterm

Problem 1 $\quad y_n - y_{n-2} = \frac{h}{3}\left[f_n - 3f_{n-1} + 2f_{n-2}\right]$

(a) This is an implicit method since $f_n = f(t_n, y_n)$.

(b) The associated polynomials to this method are:

$$p(z) = z^2 - 1$$
$$q(z) = \frac{1}{3}\left[z^2 - 3z + 2\right]$$

- This method is stable since the roots of $p(z)$ are ± 1 (simple roots with $|z|=1$)
- This method is NOT consistent since:

$$p(1) = 0$$
$$p'(z) = 2z \qquad p'(1) = 2$$
$$q(1) = \frac{1}{3}\left[1 - 3 + 2\right] \overset{*}{=} 0$$

- Thus the method is not convergent, since convergent $\Leftrightarrow$ (stable & consistent)

Problem 2

(a) By differentiating (2) we get:

$$\frac{\partial y''}{\partial z} = \frac{\partial}{\partial z} f(t,y,y') \underset{\substack{\uparrow \\ \text{CHAIN} \\ \text{RULE}}}{=} \frac{\partial y}{\partial z} f_y(t,y,y') + \frac{\partial y'}{\partial z} f_{y'}(t,y,y')$$

Since partial derivatives commute (for smooth enough $y(t;z)$):

$$u'' = u f_y(t,y,y') + u' f_{y'}(t,y,y')$$

Differentiating boundary conditions:

$$u(a) = \frac{\partial y(a;z)}{\partial z} = \frac{\partial}{\partial z}(\alpha) = 0$$

$$u'(a) = \frac{\partial y'(a;z)}{\partial z} = \frac{\partial}{\partial z}(z) = 1$$

$\rightsquigarrow$ we get desired IVP.

(b) $\phi(z) = y(b;z) - \beta$

$$\phi'(z) = \frac{\partial \phi}{\partial z}(z) = \frac{\partial y}{\partial z}(b;z) = u(b)$$

(c) To solve BVP (1) we need to be able to transform (2) and the variational equation for u into a system of first order ODE's. We achieve this by using the auxiliary variables:

$$w_1 = y$$
$$w_2 = y'$$
$$w_3 = u$$
$$w_4 = u'$$

Therefore we have a system:

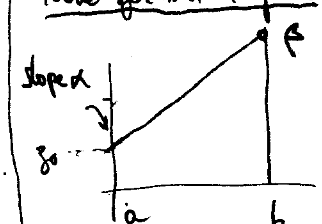
$$\text{(SYS)} \quad \begin{cases} \underline{w}' = \underline{f}(t, \underline{w}) \\ \underline{w}(a) = \underline{\alpha}(z) \end{cases}$$

where $\underline{w} = (w_1, w_2, w_3, w_4)$

and: $$\underline{f}(t, \underline{w}) = \begin{bmatrix} w_2 \\ f(t, w_1, w_2) \\ w_4 \\ w_3 f_y(t, w_1, w_2) + w_4 f_{y'}(t, w_1, w_2) \end{bmatrix}$$

$$\underline{\alpha}(z) = \begin{pmatrix} z \\ \alpha \\ 1 \\ 0 \end{pmatrix}$$

Note for initial guess:



$$\frac{\beta - z_0}{b-a} = \alpha$$

$$\Rightarrow \boxed{z_0 = \beta - \alpha(b-a)}$$

Thus the whole algorithm looks as follows:

z_0 = initial guess ; $i = 0$

Solve for $w^{(0)}$ in (SYS) with I.C. $\underline{\alpha}(z_0)$

$\phi_0 = w_1^{(0)}(b) - \beta$

while $\phi_i >$ TOL AND $i \leq$ maxit

$$z_{i+1} = z_i - \frac{(w_1^{(i)}(b) - \beta)}{w_3^{(i)}(b)}$$

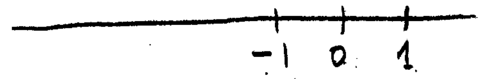
Solve for $w^{(i+1)}$ in (SYS) with I.C. $\underline{\alpha}(z_{i+1})$

$\phi_{i+1} = w_1^{(i+1)}(b) - \beta$

$i = i+1$

Problem 3

(a) We need to devise an integration formula that is exact for polynomials up to degree 2:



$$\int_0^1 p(t)\,dt = A\,p(1) + B\,p(0)$$

Using basis $p_0(t) = 1$
$p_1(t) = t$

we get system:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \Rightarrow B = \frac{1}{2},\ A = \frac{1}{2}$$

Thus we get A-M:

$$\boxed{y_{n+1} = y_n + \frac{h}{2}\left[f_{n+1} + f_n\right]}$$ (implicit trapezoidal rule)

(b) Written in the general form our method's coefficients are:

$a_1 = 1$ $\quad b_1 = \frac{1}{2}$
$a_0 = -1$ $\quad b_0 = \frac{1}{2}$

therefore:

$$d_0 = a_1 + a_0 = 0$$

$$d_1 = (0 \cdot a_0 - b_0) + \left(\frac{1}{1!}a_1 - \frac{1^0}{0!}b_1\right)$$

$$= -\frac{1}{2} + 1 - \frac{1}{2} = 0$$

$$d_2 = (0 \cdot a_0 - 0 b_0) + \frac{1^2}{2!}a_1 - \frac{1^2}{1!}b_1$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

$$d_3 = (0 \cdot a_0 - 0 b_0) + \frac{1^3}{3!}a_1 - \frac{1^2}{2!}b_1$$

$$= \frac{1}{6} - \frac{1}{4} = -\frac{1}{12} \neq 0$$

Therefore the method is of order 2 and the local truncation error is:

$$-\frac{1}{12} h^3 y^{(3)}(t_{n-1}) + O(h^4).$$

Problem 4

(a) $$y(t+h) = y(t) + h y'(t) + \frac{h^2}{2} y''(t) + O(h^3)$$
$$= y + hf + h^2 \frac{df}{dt} + O(h^3)$$

and $$\frac{d f(t,y)}{dt} = f_t(t,y) + y' f_y(t,y)$$
$$= f_t + f f_y.$$

Thus:
$$\boxed{y(t+h) = y + hf + \frac{h^2}{2}[f_t + f f_y] + O(h^3)}$$

(b) $$y(t+h) = y + w_1 hf + w_2 h f(t+\alpha h, y+\beta h f) + O(h^3)$$
$$= y + w_1 hf + w_2 h \left[f + \alpha h f_t + \beta h f f_y + O(h^2) \right] + O(h^3)$$
$$= y + \underbrace{w_1 hf + w_2 hf}_{= h(w_1+w_2)f} + w_2 h^2 (\alpha f_t + \beta f f_y) + O(h^3)$$

(c) For the method to be order 2, it has to match Taylor series from (a):

$w_1 + w_2 = 1$ (h term)

$w_2 \alpha = \frac{1}{2}$ (h^2 term)

$w_2 \beta = \frac{1}{2}$

(d) If $w_1 = \frac{1}{4}$, $w_2 = \frac{3}{4}$ and: $\alpha = \frac{1}{2}\frac{4}{3} = \frac{2}{3}$, $\beta = \frac{1}{2}\frac{4}{3} = \frac{2}{3}$

Thus the method is:

y_0 = initial condition

for $i = 0 .. n$

$F_1 = h f(t_i, y_i)$

$F_2 = h f(t_i + \frac{2}{3}h, y_i + \frac{2}{3}F_1)$

$y_{i+1} = y_i + \frac{1}{4}F_1 + \frac{3}{4}F_2$