Problem 1

\( y_n - y_{n-2} = \frac{h}{3} [f_n - 3f_{n-1} + 2f_{n-2}] \)

(a) This is an implicit method since \( f_n = F(t_n, y_n) \).

(b) The associated polynomials to this method are:

\[
\begin{align*}
p(3) &= 8^2 - 1 \\
q(3) &= \frac{1}{3} [8^2 - 3(8) + 2] \\
\end{align*}
\]

This method is stable since the root of \( p(3) \) are \( \pm 1 \) (Simple root with \( |q(3)| \leq 1 \)).

This method is not consistent since:

\[
\begin{align*}
p(4) &= 0 \\
p'(3) &= 2 \times 8 \\
p''(3) &= \frac{2}{3} \\
q(1) &= \frac{1}{3} [4 - 3 + 2] = 0 \\
\end{align*}
\]

Thus the method is not convergent, since convergent \( \Rightarrow \) (stable & consistent)

Problem 2

(a) By differentiating (2) we get:

\[
\frac{\partial y''}{\partial y} = \frac{3}{3} \frac{\partial}{\partial y} \frac{\partial y}{\partial y} \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \frac{\partial y}{\partial y} \frac{\partial}{\partial y} \frac{\partial y}{\partial y} \\
\text{Chain Rule}
\]

Since partial derivatives commute (for smooth enough \( y(t; \lambda) \)):

\[
y'' = y y' + y' y''
\]

Differentiating boundary conditions:

\[
\begin{align*}
\mu(a) &= \frac{\partial y(a; \lambda)}{\partial \lambda} = \frac{\partial}{\partial \lambda} (a) = 0 \\
\mu'(a) &= \frac{\partial y'(a; \lambda)}{\partial \lambda} = \frac{\partial}{\partial \lambda} (\lambda) = 1 \\
\end{align*}
\]

\( \Rightarrow \) we get desired IVP.
(b) \( \phi(x) = y(x+b) - \beta \)
\[ \phi'(x) = \frac{\partial \phi}{\partial x} = y'(x+b) = \mu_G(x+b) \]

(c) To solve BVP (1) we need to be able to transform (2) and the variational equation for \( u \) into a system of first order ODE's. We achieve this by using the auxiliary variables:

\[ \begin{align*}
    w_1 &= y \\
    w_2 &= y' \\
    w_3 &= \mu \\
    w_4 &= \mu'
\end{align*} \]

Therefore we have a system:

\[ \begin{align*}
    \frac{dw_1}{dt} &= f(t, w) \\
    \frac{dw_2}{dt} &= f_1(t, w) \\
    \frac{dw_3}{dt} &= f_2(t, w) \\
    \frac{dw_4}{dt} &= f_3(t, w)
\end{align*} \]

where \( w = (w_1, w_2, w_3, w_4) \)

and:

\[ f(t, w) = \begin{bmatrix} w_2 \\
    f_1(t, w_1, w_2) \\
    w_3 \\
    f_2(t, w_1, w_2) + w_4 f_3(t, w_1, w_2) \end{bmatrix} \]

\[ \alpha(x) = \begin{pmatrix}
    \beta \\
    \omega \\
    0 \\
    0
\end{pmatrix} \]

Thus the whole algorithm looks as follows:

\[ z_0 = \text{Initial guess} ; i = 0 \]

\[ \text{Solve for } w^{(0)} \text{ in (sys) with I.C. } \alpha(z_0) \]

\[ \phi_0 = w_1^{(0)}(b) - \beta \]

\[ \text{while } \phi_i > \text{Tol AND } i \leq \text{maxit} \]

\[ z_{i+1} = z_i - \frac{\phi_i}{w_3^{(i)}(b)} \\
\]

\[ \text{Solve for } w^{(i+1)} \text{ in (sys) with I.C. } \alpha(z_{i+1}) \]

\[ \phi_{i+1} = w_1^{(i+1)}(b) - \beta \]

\[ i = i + 1 \]

Note for initial guess:

\[ \text{Slope: } \alpha = \frac{\beta - \delta^*}{b-a} \]

\[ \Rightarrow z_0 = \beta - \alpha (b-a) \]
Problem 3

(a) We need to devise an integration formula that is exact for polynomials up to degree 2:

\[
\int_0^1 p(t) \, dt = A p(1) + B p(0)
\]

Using basis \( p_0(t) = 1 \)
\( p_1(t) = t \)
we get system:

\[
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
\frac{1}{2}
\end{bmatrix}
\Rightarrow 
B = \frac{1}{2}, 
A = \frac{1}{2}
\]

Thus we get A-M:

\[
y_{n+1} = y_n + \frac{h}{2} \left[ f_{n+1} + f_n \right]
\] (implicit trapezoidal rule)

(b) Written in the general form our method's coefficients are:

\[
\begin{align*}
a_1 &= 1 \\
b_1 &= \frac{1}{2} \\
a_0 &= -1 \\
b_0 &= \frac{1}{2}
\end{align*}
\]

Therefore:

\[
\begin{align*}
d_0 &= a_1 + a_0 = 0 \\
d_1 &= (0 \cdot a_0 - b_0) + \left( \frac{1}{2} a_1 - \frac{1}{2} b_1 \right) \\
      &= \frac{1}{2} + 1 - \frac{1}{2} = 0 \\
d_2 &= (0 \cdot a_0 - b_0) + \frac{1}{2} a_1 - \frac{12}{12} b_1 \\
      &= \frac{1}{2} - \frac{1}{2} = 0 \\
d_3 &= (0 \cdot a_0 - b_0) + \frac{1}{2} a_1 - \frac{1}{2} b_1 \\
      &= \frac{1}{2} - \frac{1}{4} = -\frac{1}{12} \neq 0
\end{align*}
\]

Therefore the method is of order 2 and the local truncation errors:

\[
\frac{1}{12} \, h^2 \, y^{(3)}(t_{n-1}) + O(h^4).
\]
Problem 4

(a) \( y(t+h) = y(t) + hf + \frac{h^2}{2} \frac{d^2 y}{dt^2} + O(h^3) \)

\[ = y + hf + \frac{h^2}{2} \frac{df}{dt} + O(h^3) \]

and \( \frac{df}{dt} = f_t + f_y. \)

Thus:

\( y(t+h) = y + hf + \frac{h^2}{2} [f_t + f_y] + O(h^3) \)

(b) \( y(t+h) = y + w_1 hf + w_2 h^2 \frac{df}{dt} + O(h^3) \)

\[ = y + w_1 hf + w_2 h^2 \left[ f_t + \alpha hf + \beta h^2 f_y + O(h^2) \right] \]

\[ = y + w_1 hf + w_2 h^2 f_t + w_2 h^2 \left( \alpha f_t + \beta f_y \right) + O(h^3) \]

\[ = y + \underbrace{w_1 hf + w_2 h^2 f_t}_R + \underbrace{w_2 h^2 \left( \alpha f_t + \beta f_y \right)}_{R^2} + O(h^3) \]

(c) For the method to be order 2, it has to match Taylor's expansion (a):

\[ w_1 + w_2 = 1 \quad (R \text{ term}) \]
\[ w_2 \alpha = \frac{1}{2} \quad (R^2 \text{ term}) \]
\[ w_2 \beta = \frac{1}{2} \]

(d) If \( w_1 = \frac{1}{4}, \ w_2 = \frac{3}{4} \) and:

\[ \alpha = \frac{1}{2} \quad \beta = \frac{1}{3} \]

Thus the method is:

\[ y_0 = \text{initial condition} \]

for \( n = 0 \ldots m \)

\[ F_1 = hf(t_i, y_i) \]
\[ F_2 = hf(t_i + \frac{3}{4}h, y_i + \frac{2}{3}F_1) \]

\[ y_{i+1} = y_i + \frac{1}{4} F_1 + \frac{3}{4} F_2 \]