MATH 5620 – NUMERICAL ANALYSIS II
PRACTICE MIDTERM EXAM

Problem 1. Consider the multistep method:

$$y_n - y_{n-2} = \frac{h}{3}[f_n - 3f_{n-1} + 2f_{n-2}]$$

(a) Is this method implicit or explicit?
(b) Is this method convergent, stable and/or consistent? Justify your answer.

Problem 2. The goal of this problem is to design a non-linear shooting method for solving a non-linear BVP with mixed type boundary conditions:

$$\begin{cases} y'' = f(t, y, y'), \text{ for } t \in [a, b], \\ y'(a) = \alpha, \\ y(b) = \beta. \end{cases} \tag{1}$$

Let $y = y(t; z)$ be the solution to the IVP

$$\begin{cases} y'' = f(t, y, y'), \text{ for } t \in [a, b], \\ y(a) = z, \\ y'(a) = \alpha. \end{cases} \tag{2}$$

and let $\phi(z) = y(b; z) - \beta$.

(a) Let $u(t) = \frac{\partial y(t;z)}{\partial z}$. By differentiating (2) with respect to z, show that u solves

$$\begin{cases} u'' = uf_y(t, y, y') + u'f_{y'}(t, y, y'), \text{ for } t \in [a, b], \\ u(a) = 1 \\ u'(a) = 0. \end{cases}$$

(b) Show that $\phi'(z) = u(b)$.
(c) Assuming the availability of a routine for solving systems of first order equations, write pseudocode for solving (1), based on Newton's method for finding z such that $\phi(z) = 0$.

Problem 3.

(a) Using the method of undetermined coefficients derive the second order Adams-Moulton formula of the form

$$y_{n+1} = y_n + h[Af_{n+1} + Bf_n]$$

(b) Recall that for a linear multistep method of the form

$$a_k y_n + a_{k-1} y_{n-1} + \cdots + a_0 y_{n-k} = h\left[b_k f_n + b_{k-1} f_{n-1} + \cdots + b_0 f_{n-k}\right]$$

we associate the linear functional

$$Ly = \sum_{i=0}^{k} [a_i y(ih) - hb_i y'(ih)]$$

which has Taylor expansion at $t = 0$:

$$Ly = d_0 y(0) + d_1 y'(0) + d_2 h^2 y''(0) + \cdots$$

where

$$d_0 = \sum_{i=0}^{k} a_k$$

$$d_j = \sum_{i=0}^{k} \left(\frac{i^j}{j!} a_i - \frac{i^{j-1}}{(j-1)!} b_i \right), \ j \geq 1.$$

Verify that this is an order 2 method and find the local truncation error, i.e. find the constant C for which

$$y(t_n) - y_n = C h^3 y^{(3)}(t_{n-1}) + \mathcal{O}(h^4),$$

where the previous values y_{n-1}, y_{n-2}, $\ldots$ are assumed exact.

Problem 4. Consider the IVP

$$\begin{cases} y' = f(t, y) \\ y(a) = \alpha. \end{cases} \tag{3}$$

(a) Write the first 3 terms of the Taylor series for $y(t + h)$ expanding around t. Your series should be in terms of h, f and its partial derivatives. The residual should be $\mathcal{O}(h^3)$.

(b) Recall the general formula for a second-order Runge-Kutta method:

$$y(t + h) = y + w_1 h f + w_2 h f(t + \alpha h, y + \beta h f) + \mathcal{O}(h^3), \tag{4}$$

where $y \equiv y(t)$ and $f \equiv f(t, y)$. Use the two variable Taylor expansion

$$f(t + hs, y + hv) = f(t, y) + hs f_t(t, y) + hv f_y(t, y) + \mathcal{O}(h^2)$$

to express (4) in terms of y, f, $f_t \equiv f_t(t, y)$ and $f_y \equiv f_y(t, y)$.

(c) What conditions should w_1, w_2, α and β satisfy in order for the method to be second order?

(d) Write down the particular Runge-Kutta method of order 2 with $w_1 = 1/4$.