MATH 5610 MIDTERM SOLUTIONS

Problem 1: $f(h) = \dfrac{e^h - (1+h)}{h^2}$

(a) $e^h = 1 + h + \frac{h^2}{2} + \mathcal{O}(h^3)$

$$\Rightarrow f(h) = \frac{1 + h + \frac{h^2}{2} + \mathcal{O}(h^3) - (1+h)}{h^2}$$

$$= \frac{1}{2} + \frac{h}{3} + o(h)$$

$$\Rightarrow \lim_{h \to 0} f(h) = \frac{1}{2}$$

(b) $f(h) = \frac{1}{2} + o(1)$

$$= \frac{1}{2} + \mathcal{O}(h)$$

Problem 2

(a) $\dfrac{n^2+1}{n^3} = \dfrac{1}{n} + \dfrac{1}{n^3} = o(1)$ true.

(b) $\dfrac{2^n}{n^{10}} = \mathcal{O}\left(\frac{1}{n}\right)$ is false since:

$$\lim_{n \to \infty} \frac{2^n}{n^{10}} = \infty.$$

(c) $\dfrac{1}{ne^n} = \mathcal{O}\left(\frac{1}{n}\right)$ true.

(d) $\sin\left(\frac{1}{n}\right) = o\left(\frac{1}{n}\right)$ false since $\displaystyle\lim_{n \to 0} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$

Problem 3

(a) $$x = 2^{-3} + 2^{-24} + 2^{-28}$$
$$= 2^{-3}(1 + 2^{-21} + 2^{-25})$$
$$= (1.\underbrace{0\ldots0}_{20 \text{ zeros}}10001)_2 \times 2^{-3}$$

(b) $$x_- = (1.\underbrace{0\ldots0}_{20 \text{ zeros}}100)_2 \times 2^{-3}$$
$$x_+ = (1.\underbrace{0\ldots0}_{20 \text{ zeros}}101)_2 \times 2^{-3}$$

(c) $fl(x) = x_-$

(d) $\varepsilon = 2^{-23}$ (one in last position)

(e) $$\frac{|x - fl(x)|}{|x|} = \frac{2^{-28}}{2^{-3} + 2^{-24} + 2^{-28}} \leq 2^{-25} \leq \varepsilon = 2^{-23}$$

Problem 4 $p(z) = z^4 - 6z^3 + 11z^2 - 3z - 5$

	1	-6	11	-3	-5
2		2	-8	6	6
	1	-4	3	3	$\boxed{1}$
2		2	-4	-2	
	1	-2	-1	$\boxed{1}$	
2		2	0		
	1	0	$\boxed{-1}$		
2		2			
	$\boxed{1}$	$\boxed{2}$			

$$\boxed{p(z) = 1 + (z-2) - (z-2)^2 + 2(z-2)^3 + (z-2)^4}$$

Problem 5

x	1	2	3	4
$f(x)$	1	2	-1	4

(a) $$p(x) = 1\,\frac{(x-2)(x-3)}{(1-2)(1-3)} + 2\,\frac{(x-1)(x-3)}{(2-1)(2-3)}$$

$$+ (-1)\,\frac{(x-1)(x-2)}{(3-1)(3-2)}$$

(b)

1	1	1	-2	2
2	2	-3	4	
3	-1	5		
4	4			

$$p(x) = 1 + (x-1) - 2(x-1)(x-2) + 2(x-1)(x-2)(x-3)$$

(c) $$f(t) = p(t) + \frac{f^{(5)}(\xi)}{5!}(t-1)(t-2)(t-3)(t-4)$$

Should be f^{(4)}(\xi) / 4!

where $\xi \in [1,4]$

Problem 6

(a) $f(x_*) = 0$, $f'(x_*) \neq 0$

$\Leftrightarrow$ $f(x) = (x - x_*)g(x)$ with $\lim_{x \to x_*} g(x) \neq 0$.

$\Leftrightarrow$ x_* is a simple root of f.

(b) $x_{n+1} = g(x_n)$, $g(x) = x - \dfrac{f(x)}{f'(x)}$

(c) $$g'(x) = 1 - \frac{(f'(x))^2 - f''(x)f(x)}{(f'(x))^2}$$

$$= \frac{f''(x)f(x)}{(f'(x))^2}$$

$\Rightarrow$ $g'(x_*) = 0$ since $f(x_*) = 0$.

(d) Using notation $f \equiv f(x)$, $f_* \equiv f(x_*)$, ...
we get:

$$g''(x) = \frac{(f'''f + f''f')(f')^2 - 2f''f'f''f}{(f')^4}$$

$$\Rightarrow g''(x_*) = \frac{f''_* f'_* (f'_*)^2}{(f'_*)^4} \quad (\text{since } f_* = 0)$$

(e) Convergence rate is quadratic since this is a fixed point iteration with $g'(x) = 0$, $g''(x) \neq 0$ (in general)

(f) To get faster than quadratic convergence, we need $f''(x_*) = 0$ (cubic or better)

<u>Problem 7</u> RHS is interp. poly of $p(x)$ at nodes $x_0 \dots x_n$ and is a poly of degree $\leq n$.
By definition both poly agree at $n+1$ distinct nodes and therefore must be equal.