MATH 561 / 6860
PRACTICE MIDTERM
SOLUTIONS

Problem 1 Let $\alpha_n \to 0$, $x_n = \mathcal{O}(\alpha_n)$ and $y_n = \mathcal{O}(\alpha_n)$.

Then there are constants $c_1, c_2 > 0$ s.t.

$|x_n| \leq c_1 |\alpha_n|$ and $|y_n| \leq c_2 |\alpha_n|$ for n sufficiently large.

Therefore:

$$|x_n y_n| \leq \underbrace{c_1 c_2 |\alpha_n|}_{\varepsilon_n} |\alpha_n|$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Thus $\boxed{|x_n y_n| = o(\alpha_n)}$.

Problem 2

(a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

my favourite way of remembering this is starting from geometric series:

$$\frac{1}{1+x} = (\ln(1+x))' = 1 - x + x^2 - x^3 + x^4 + \cdots$$

To find error term we need $f^{(n)}(x)$, where $f(x) = \ln(1+x)$.

$f'(x) = (1+x)^{-1}$

$f''(x) = -(1+x)^{-2}$

$f^{(3)}(x) = +2(1+x)^{-3}$

$\vdots$

$f^{(n)}(x) = (n-1)!\,(-1)^{n-1}(1+x)^{-n}$

Thus :

$$\boxed{\ln(1+x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} + \frac{1}{(n+1)!} \underbrace{n!(-1)^n(1+\xi)^{-(n+1)}}_{f^{(n+1)}(\xi)} x^{n+1}}$$

for some ξ between x and 0.

(b) The approx is:

$$\ln(2) \simeq \sum_{k=1}^{10} \frac{(-1)^{k-1}}{k}$$

The error is:

$$\boxed{\left|\ln(2) - \sum_{k=1}^{10} \frac{(-1)^{k-1}}{k}\right| \leqslant \frac{1}{11} \frac{1}{|1+\xi|^{11}} \leqslant \frac{1}{11}} \text{ for } \xi \in (0,1)$$

since $\underline{1 \leqslant |1+\xi| \leqslant 2 \text{ for } \xi \in (0,1).}$

Problem 3:

(a) $x = 2^6 + 2^{-16} + 2^{-19} = 2^6(1 + 2^{-22} + 2^{-25})$

$$= (\underbrace{1.\underbrace{0 \cdots 0}_{21\text{ zeros}}1001}_{23+1\text{ bits stored}}) \times 2^6$$

(b) $x_+ = (1.\underbrace{0 \cdots 0}_{21}11) \times 2^6$

$x_- = (1.\underbrace{0 \cdots 0}_{21}10) \times 2^6$

(c) $fl(x) = x_-$ which is closest to x.

(d) $\varepsilon = 2^{-23}$

(e)

$$x - fl(x) = 2^{-19}$$

$$\frac{|x - fl(x)|}{x} = \frac{2^{-19}}{2^{6}(1 + 2^{-22} + 2^{-25})}$$

$$\leq 2^{-25} \leq \varepsilon$$

Problem 4

$$x_{n+1} = x_n - \frac{f_n f_n'}{(f_n')^2 - (f_n f_n'')/2}.$$

Applying Newton's method to $g = f/\sqrt{f'}$ we get:

$$x_{n+1} = x_n - \frac{f_n/\sqrt{f_n'}}{(f_n/\sqrt{f_n'})'}$$

and

$$\left(\frac{f_n}{\sqrt{f_n'}}\right)' = \frac{f_n'\sqrt{f_n'} - f_n f_n''/(2\sqrt{f_n'})}{f_n'}$$

$$= \frac{f_n' - f_n f_n''/(2f_n')}{\sqrt{f_n'}}$$

thus iteration is:

$$x_{n+1} = x_n - \frac{f_n}{\cancel{\sqrt{f_n'}}} \times \frac{\cancel{\sqrt{f_n'}}}{f_n' - f_n f_n''/(2f_n')}$$

$$\boxed{= x_n - \frac{f_n f_n'}{(f_n')^2 - f_n f_n''}}$$

Problem 5

$$p(z) = z^4 + 2z^3 + 3z^2 + 3z + 2$$

(a) Horner's method

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 3 & 2 \\ 2 & & 2 & 8 & 22 & 50 \\ \hline & 1 & 4 & 11 & 25 & \boxed{52} \end{array}$$

$$\boxed{p(2) = 52}$$

(b)

$$\begin{array}{c|cccc} & 1 & 4 & 11 & 25 \\ 2 & & 2 & 12 & 46 \\ \hline & 1 & 6 & 23 & \boxed{71} \end{array}$$

$$\boxed{p'(2) = 71}$$

(c) $$\boxed{p(z) = (z-2)(z^3 + 4z^2 + 11z + 25) + 52}$$

Problem 6

x	-1	0	1	2
$f(x)$	-1	0	3	-1

(a) $$\boxed{P_2(x)} = \frac{x(x-1)}{(-1)(-1-1)} \times (-1) + \frac{(x+1)(x-1)}{(0+1)(0-1)} \times 0$$

$$+ \frac{x(x+1)}{1(1+1)} \times 3$$

$$\boxed{= -\frac{1}{2} x(x-1) + \frac{3}{2} x(x+1)}$$

(good idea to double check!)

(b) Divide differences table:

-1	-1	1	1	-3/2
0	0	3	-7/2	
1	3	-4		
2	-1			

Thus:

$$P_3(x) = \underbrace{-1 + (x+1) + x(x+1)}_{\substack{\text{can double check by seen that} \\ \text{this is } P_2(x).}} - \frac{3}{2} x(x+1)(x-1)$$

(c) $$\boxed{f(t) - p(t) = \frac{1}{4!} f^{(4)}(\xi)(t+1)t(t-1)(t-2)}$$

for some ξ in $[-1, 2]$.

alternatively the error is given using dir. differences by:

$$\boxed{f(t) - p(t) = f[x_0, x_1, x_2, x_3, t](t+1)t(t-1)(t-2)}$$

Problem 7 The polynomials

$$p_n(x) \text{ and } q(x) + \frac{x - x_n}{x_n - x_0}(q(x) - p_{n-1}(x))$$

are of degree $\leq n$. To show they are equal it suffices to compare them at the $n+1$ distinct points $x_0, \dots, x_n$.

We have:

$$p_n(x_0) = f(x_0)$$
$$\Updownarrow$$
$$q(x_0) + \frac{x_0 - x_n}{x_n - x_0}\left(q(x_0) - \underbrace{p_{n-1}(x_0)}_{= f(x_0)}\right) = f(x_0)$$

$$p_n(x_n) = f(x_n)$$
$$\Updownarrow$$
$$\underbrace{q(x_n)}_{= f(x_n)} + \frac{x_n - x_n}{x_n - x_0}\left(\quad \right) = f(x_n)$$

And for $j = 1, \dots, n-1$:

$$p_n(x_j) = f(x_j)$$
$$\Updownarrow$$
$$\underbrace{q(x_j)}_{= f(x_j)} + \frac{x_j - x_n}{x_n - x_0}\underbrace{\left(q(x_j) - p_{n-1}(x_j)\right)}_{= 0} = f(x_j)$$

QED.