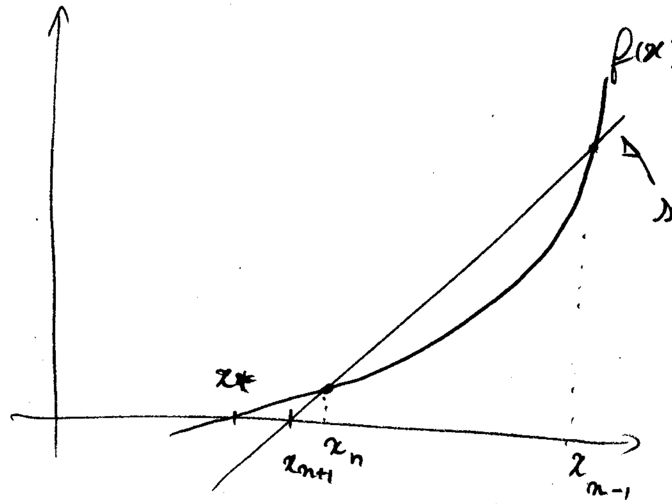


Graphical interpretation of secant method

§ 2.2 Fixed point iteration

Recall Newton's method iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This can be rewritten in the form:

$$x_{n+1} = F(x_n) \qquad (*)$$

where $F(x) = x - \dfrac{f(x)}{f'(x)}$.

An iteration of the form $(*)$ is called <u>fixed point iteration</u>, since if $x_n \to x_*$ and $F(x)$ is continuous, then x_* must be a fixed point of $F(x)$, i.e. $x^* = F(x_*)$.

This is easily seen from def on continuity:

$$F(x_*) = F\left(\lim_{n\to\infty} x_n\right) \underset{\substack{\uparrow \\ F\text{ cont.}}}{=} \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_{n+1} = x_* .$$

Of course not all iterations of the form (*) converge, an easy counterexample being:

$$F(x) = 2x \quad \rightsquigarrow x_n = 2^n x_0 \text{ which diverges!}$$

The next theorem is very powerful and ubiquitous in applied math and gives a condition under which F.P. exists and is unique and iteration (*) converges to that particular fixed point.

But first recall the def: f is Lipschitz continuous with Lipschitz constant L ;

$$\forall x, y \qquad |f(x) - f(y)| \leq L\, |x - y|$$

Theorem (Contractive mapping theorem)

Let C be a closed and bounded set of $\mathbb{R}$.
If F is Lipschitz continuous with Lipschitz constant $L < 1$, then F has a unique fixed point $x_* \in C$. Moreover iteration (*) converges to x_* regardless of initial iterate $x_0 \in C$.

proof:
$$|x_n - x_{n-1}| = |F(x_{n-1}) - F(x_{n-2})|$$
$$\leq L\, |x_{n-1} - x_{n-2}|$$

Thus:
$$\begin{aligned} |x_n - x_{n-1}| &\leq L\, |x_{n-1} - x_{n-2}| \\ &\leq L^2\, |x_{n-2} - x_{n-3}| \\ &\vdots \\ &\leq L^{n-1}\, |x_1 - x_0| \end{aligned}$$

Now we can write x_n w/ telescoping series:

$$x_n = x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1})$$
$$= x_0 + \sum_{k=1}^{n} (x_k - x_{k-1})$$

$\{x_n\}$ converges $\Leftrightarrow$ $\sum_{k=1}^{\infty} x_k - x_{k-1}$ converges.

To show that the latter series converges it suffices to show that:

$$\sum_{k=1}^{\infty} |x_k - x_{k-1}| \quad \text{converges.}$$

This is true since $|x_k - x_{k-1}| \leq L^{k-1} |x_1 - x_0|$

which means series is dominated by a geometric series:

$$\sum_{k=1}^{\infty} |x_k - x_{k-1}| \leq \sum_{k=1}^{\infty} L^{k-1} |x_1 - x_0| = \frac{1}{1-L} |x_1 - x_0| .$$

Now let $x_* = \lim_{n \to \infty} x_n = x_0 + \lim_{n \to \infty} \sum_{k=1}^{n} x_k - x_{k-1}$

By continuity of F, we have $x_* = F(x_*)$, so the limit is a fixed point.

What about uniqueness? Assume there are two fixed points x_* and y_*, with $x_* \neq y_*$ for contradiction. Then:

$$|x_* - y_*| = |F(x_*) - F(y_*)| \leq L |x_* - y_*|$$
$$< |x_* - y_*|$$

contradiction.

$\Rightarrow$ $x_* = y_*$ and fixed point is unique.

One other way of proving contractive mapping theorem is to invoke Cauchy's criterion.

A Cauchy sequence $\{x_n\}$ is a sequence s.t.

$\forall \varepsilon > 0 \; \exists N_0 \quad \forall n, m \geq N_0 \qquad |x_n - x_m| < \varepsilon$

The real line $\mathbb{R}$ is an example of a complete metric space where every Cauchy sequence converges inside the space ($\mathbb{R}$)

So to show convergence of F.P. iteration we can also use Cauchy criterion: $(n \geq m \geq N_0)$

$$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + x_{m+1} + x_{m+1} - x_m|$$

$$\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$$

$$\leq L^{n-1}|x_1 - x_0| + L^{n-2}|x_1 - x_0| + \cdots + L^m |x_1 - x_0|$$

$$= L^m |x_1 - x_0| \left(1 + L + L^2 + \cdots + L^{n-m-1}\right)$$

$$= L^m |x_1 - x_0| \frac{1 - L^{n-m}}{1 - L}$$

$$\leq \frac{L^{N_0}}{1-L} |x_1 - x_0|$$

$\rightarrow$ can be made as small as possible provided N_0 is large enough.

Example: Show that $\{x_n\}$ is convergent:

$$\begin{cases} x_0 = -15 \\ x_{n+1} = 3 - \frac{1}{2}|x_n| \end{cases}$$

$F(x) = 3 - \frac{1}{2}|x|$ is a contraction since:

triangle ineq.

$$|F(x)-F(y)| \leq \frac{1}{2}\,\big||x|-|y|\big| \leq \frac{1}{2}|x-y|$$

$\Rightarrow$ iteration must converge to the unique fixed point of F, which is 2.

Error analysis

Consider the F.P. iteration:

$$\begin{cases} x_0 = \text{given} \\ x_{n+1} = F(x_n) \end{cases}$$

with $x_n \to x_*$ as $n \to \infty$.

Assuming $F \in C^1$; we can apply MVT

$$x_{n+1} - x_* = F(x_n) - F(x_*) = F'(\xi_n)(x_n - x_*),$$

for some ξ_n between x_n and x_*.

Thus having $|F'(\xi_n)| < 1$ guarantees convergence.
(you can see that this cdto $\Rightarrow$ Lipschitz constant < 1)

Since close to fixed point $F'(\xi_n) \approx F'(x_*)$ we could ask what happens in the extreme case where

$$F'(x_*) = 0 \;?$$

We shall see what happens in the more general case:

$$\begin{aligned} &F^{(k)}(x_*) = 0 \quad \text{for } 1 \leq k < q \\ \text{but } &F^{(q)}(x_*) \neq 0 \end{aligned} \qquad (**)$$

We use Taylor's theorem again:

$$x_{n+1} - x_* = F(x_n) - F(x_*)$$
$$= F(x_* + x_n - x_*) - F(x_*)$$
$$= F(x_*) + (x_n - x_*)F'(x_*) + \frac{1}{2}(x_n - x_*)^2 F''(x_*) + \dots$$
$$- F(x_*)$$
$$= \sum_{k=1}^{q-1} \frac{(x_n - x_*)^k}{k!} F^{(k)}(x_*) + \frac{1}{q!}(x_n - x_*)^q F^{(q)}(\xi_n)$$

where ξ_n is between x_n and x_*.

Under assumption $(**)$ the first $q-1$ derivatives of F vanish at x_*, so:

$$x_{n+1} - x_* = \frac{1}{q!}(x_n - x_*)^q F^{(q)}(\xi_n)$$

$$\Rightarrow \quad |x_{n+1} - x_*| \leq \underbrace{\frac{|F^{(q)}(\xi_n)|}{q!}}_{\text{bounded if } F^{(q)} \text{ is continuous}} |x_n - x_*|^q$$

$$\leq M |x_n - x_*|^q$$

$$\Rightarrow \boxed{x_n \to x_* \text{ with order of convergence } q.}$$

Example We can show quadratic convergence of Newton's method using the above:

$$F(x) = x - \frac{f(x)}{f'(x)}$$

$$F'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

Since x_* fixed point of of F is a zero of f:

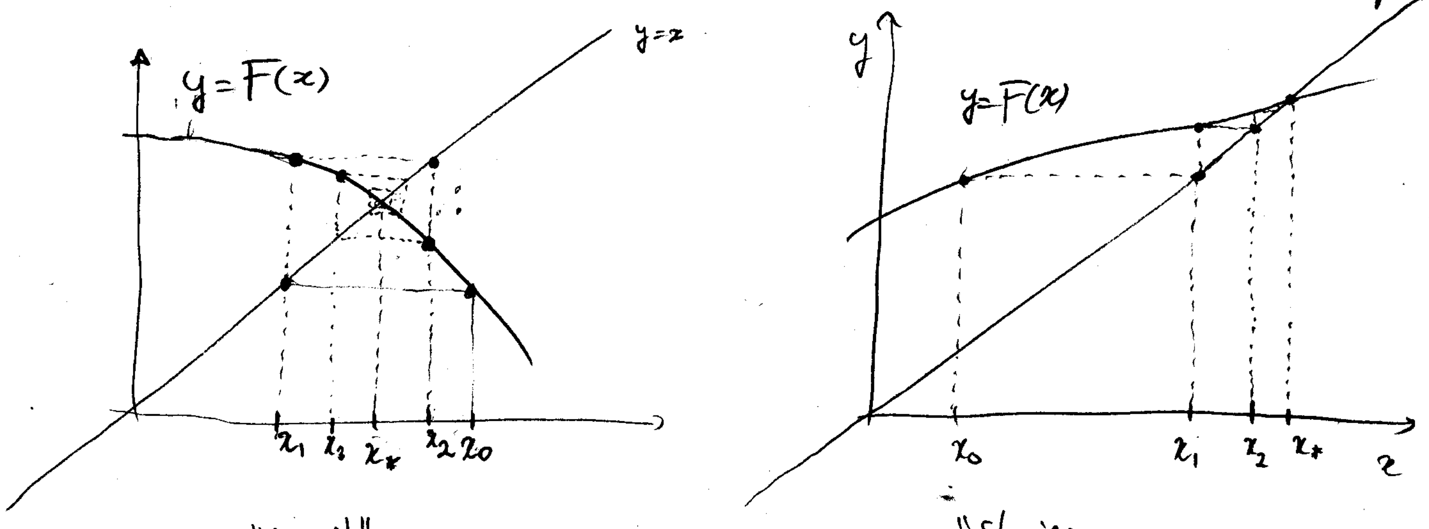
$$F'(x_*) = \frac{f(x_*)f''(x_*)}{[f'(x_*)]^2} = 0$$

$$F''(x) = \frac{(f')^2(ff''' + f'f'') - (2f''f')(ff'')}{(f')^4}$$

$$\Rightarrow F''(x_*) = \frac{f''(x_*)}{f'(x_*)} \neq 0 \quad \text{(usually)}$$

In general the order of convergence of a fixed point iteration is the first integer q for which $F^{(q)}(x_*) \neq 0$.

Graphical interpretation



"snail" "staircase"

Newton's method and multiple roots of a function

<u>Def</u> A root x_* of $f(x)$ is said to be a <u>root of multiplicity q</u> of f if we can write:

$$f(x) = (x - x_*)^q \, g(x)$$

with $\lim_{x \to x_*} g(x) \neq 0$

One can easily identify such zeros by looking at derivatives:

<u>Thm</u> $f \in C^1[a,b]$ has a simple zero in (a,b) iff $f(x_*) = 0$ but $f'(x_*) \neq 0$.

<u>proof</u>: $\underline{\Rightarrow}$: if f has a simple zero at x_* then:

$f(x_*) = 0$ and

$f(x) = (x - x_*) \, g(x)$ with $\lim_{x \to x_*} g(x) \neq 0$.

But $f \in C^1[a,b]$ so:

$$f'(x_*) = \lim_{x \to x_*} f'(x) = \lim_{x \to x_*} g(x) + (x - x_*) g'(x)$$

$$= \lim_{x \to x_*} g(x) \neq 0.$$

$\underline{\Leftarrow}$: if $f(x_*) = 0$ but $f'(x_*) \neq 0$ we can use Taylor's theorem around x_*:

$$f(x) = f(x_*) + f'(\xi(x))(x - x_*)$$

$$= (x - x_*) \, f'(\xi(x))$$

where $\xi(x)$ is some point between x and x_*.

We obviously have: $\lim_{x \to x_*} \xi(x) = x_*$

and by continuity of f':

$$\lim_{x \to x_*} f'(\xi(x)) = f'(\lim_{x \to x_*} \xi(x)) = f'(x_*) \neq 0$$

We then choose $g(x) = f'(\xi(x))$

$\Rightarrow \quad f(x) = (x - x_*)\, g(x)$

$\Rightarrow$ ~~x_*~~ is a simple zero of f.

One can easily extend this theorem (HW) to higher multiplicities:

Thm The function $f \in C^m[a,b]$ has a zero x_* of multiplicity q if and only if:

$$0 = f(x_*) = f'(x_*) = f''(x_*) = \cdots = f^{(q-1)}(x_*)$$

but $0 \neq f^{(q)}(x_*)$.

If x_* is not a simple zero of $f(x)$ then Newton's method may converge but not quadratically.

Example: $f(x) = e^x - x - 1$, $\quad f(0) = e^0 - 0 - 1 = 0$

$f'(0) = e^0 - 1 = 0$

$f''(0) = e^0 \neq 0$

0 is a zero of multiplicity 2 of $f(x)$