MATH 5610 HW1 SOLUTIONS

Problem 1 B&F 1.1.2 a, b

a) $f(x) = x - 3^{-x}$

$f(0) = -1$

$f(1) = 1 - \frac{1}{3} = \frac{2}{3}$

Since $f(x)$ is continuous and $f(0)f(1) < 0$, there must be a root in $[0,1]$.

b) $f(x) = 4x^2 - e^x$

$f(0) = -1 < 0$

$f(1) = 4 - e > 0$

for same reason as above, f must have a root in interval $[0,1]$.

Problem 2 B&F 1.1.8

$f(x) = \sqrt{x+1}$

$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}(x+1)^{-\frac{3}{2}}$

$f^{(3)}(x) = \frac{3}{8}(x+1)^{-\frac{5}{2}}$

$f(0) = 1$

$f'(0) = \frac{1}{2}$

$f''(0) = -\frac{1}{4}$

$f^{(3)}(0) = \frac{3}{8}$

Thus:

$$f(x) = P_3(x) + O(x^4),$$

where

$$\boxed{P_3(x) =} \; f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$\boxed{= 1 + \frac{x}{2} + \left(-\frac{1}{8}\right)x^2 + \frac{1}{16}x^3}$$

Now approximating $f(x)$ with $P_3(x)$ we get:

$1+x$	$f(x)$	$p(x)$	error
0.5	0.7071	0.7109	3.83×10^{-3}
0.75	0.866025	0.866211	1.86×10^{-4}
1.25	1.118034	1.118164	1.30×10^{-4}
1.75	1.3229	1.3311	8.18×10^{-3}

Problem 3 B&F 1.3.3

$$\operatorname{atan}(x) = \lim_{n \to \infty} \sum_{i=1}^{n} (-1)^{i+1} \frac{x^{2i-1}}{2i-1}$$

$$P_n(x) = \sum_{i=1}^{n} (-1)^{i+1} \frac{x^{2i-1}}{2i-1}$$

$$\left| 4P_n(x) - \pi \right| = \left| 4 \sum_{i=1}^{n} \frac{(-1)^{i+1}}{2i-1} - \pi \right| \leq \left| 4 \frac{(-1)^{n+2}}{2(n+1)-1} \right| < \varepsilon$$

we need: $\frac{2n+1}{4} > \frac{1}{\varepsilon}$ i.e. $n > \frac{1}{2}\left(\frac{4}{\varepsilon} - 1\right)$

a) $\varepsilon = 10^{-3} \Rightarrow n \geq 2000$ terms

b) $\varepsilon = 10^{-10} \Rightarrow n \geq 2 \times 10^{10}$ terms.

Problem 4

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots = 1 + o(1) = 1 + O(h)$$

$$\cos h = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \cdots = 1 + o(h) = 1 + O(h^2)$$

$$\frac{1}{1-h^4} = 1 + h^4 + h^8 + \cdots = 1 + o(h^3) = 1 + O(h^4)$$

$$1 + \sin(h^3) = 1 + h^3 - \frac{(h^3)^3}{3!} = 1 + o(h^2) = 1 + O(h^3)$$

Problem 5

$$x = 2^3 + 2^{-19} + 2^{-22}$$
$$= 2^3(1 + 2^{-22} + 2^{-25})$$
$$= (1.\underbrace{0\cdots0}_{21 \text{ zeros}}1001)_2 \times 2^3$$

If we work on single precision, the last digit cannot be represented (machine epsilon $= 2^{-23}$).

$$x_- = (1.\underbrace{0\cdots0}_{21}10)_2 \times 2^3 = 2^3 + 2^{-19}$$

$$x_+ = (1.\underbrace{0\cdots0}_{21}11)_2 \times 2^3 = 2^3(1 + 2^{-22} + 2^{-23})$$
$$= 2^3 + 2^{-19} + 2^{-20}$$

$fl(x)$ = nearest floating point to x $= x_-$

$$|x - fl(x)| = 2^{-22} \qquad \left|\frac{x - fl(x)}{x}\right| = \frac{2^{-22}}{2^3 + 2^{-19} + 2^{-22}} < \frac{2^{-22}}{2^3} = \underset{\varepsilon/2 = 2^{-24}}{\underset{\wedge}{2^{-25}}}$$