MATH 5610 HW6 SOLUTIONS

Problem 1 : see attached code and output.

Problem 2 : polynomials orthogonal wrt inner prod $(f,g) \equiv \int_0^1 f g \, dx$

$$\boxed{\phi_0(x) = 1}$$

$$\phi_1 = x\phi_1 - \frac{(x\phi_0, \phi_0)}{(\phi_0,\phi_0)} \phi_0$$

$$\Rightarrow \phi_1(x) = x - \frac{(x,1)}{(1,1)} 1 = x - \frac{(1/2)}{1} = \boxed{x - \frac{1}{2}}$$

$$\phi_2 = x\phi_1 - \frac{(x\phi_1, \phi_1)}{(\phi_1,\phi_1)}\phi_1 - \frac{(x\phi_1,\phi_0)}{(\phi_0,\phi_0)}\phi_0$$

$$\Rightarrow \phi_2(x) = x^2 - \frac{1}{2}x - \frac{(1/24)}{(1/12)}(x - 1/2) - \frac{(1/12)}{1} 1$$

$$= \boxed{x^2 - x + \frac{1}{6}}$$

$$\phi_3 = x\phi_2 - \frac{(x\phi_2, \phi_2)}{(\phi_2,\phi_2)} - \frac{(x\phi_2,\phi_1)}{(\phi_1,\phi_1)}$$

$$\Rightarrow \phi_3(x) = x^3 - x^2 + \frac{x}{6} - \frac{(1/360)}{(1/180)}(x^2 - x + 1/6) - \frac{(1/180)}{(1/12)}(x - \frac{1}{2})$$

$$= \boxed{x^3 - \frac{3}{2}x^2 + \frac{3x}{5} - \frac{1}{20}}$$

Problem 3

Using MacLaurin series; for $x \in [-1,1]$:

$$|\sin x - P_6(x)| \leq E_6$$

where

$$P_6(x) = \frac{x^5}{5!} - \frac{x^3}{3!} + x \quad \text{and} \quad E_6 = \frac{1}{7!} \approx 1.98 \times 10^{-4}.$$

- $\underline{P_5 = \text{best approx of } P_6 \text{ in } \Pi_5}$

  Since $P_6 \in \Pi_5$, $P_5 = P_6$ with 0 error on $[-1,1]$

- $\underline{P_4 = \text{best approx of } P_5 \text{ in } \Pi_4}$

$$P_4(x) = P_5(x) - \frac{T_5(x)}{2^4}\frac{1}{5!} = \frac{-5x^3}{32} + \frac{383}{384}x$$

with

$$\|P_4 - P_5\|_\infty \le \frac{1}{2^4}\frac{1}{5!} \equiv E_4 \approx 5.21 \times 10^{-4}$$

on $[-1,1]$

Thus:

$$\|\sin x - P_4\|_\infty \le E_6 + E_4 \approx 7.19 \times 10^{-4} < 0.01.$$

- $\underline{P_3 = \text{best approx of } P_4 \text{ in } \Pi_3}$

  Since $P_4 \in \Pi_3$, $P_3 = P_4$ with 0 error on $[-1,1]$.

- $\underline{P_2 = \text{best approx of } P_3 \text{ in } \Pi_2}$

$$P_2(x) = P_3(x) - \frac{T_3(x)}{2^2}\frac{5}{32}$$
$$= \frac{169}{192}x$$

with:

$$\|P_2 - P_3\|_\infty \le \frac{1}{2^2}\frac{5}{32} \equiv E_2 \approx 0.039$$

Thus $\|\sin x - P_2\|_\infty \le E_6 + E_4 + E_2 \approx 0.040 > 0.01$

So $\underline{P_3 \text{ is best approx within tolerance}}$.

Problem 4 We need to show:

$$T_i T_j = \frac{1}{2}\left[T_{i+j} + T_{i-j}\right] \quad , \; i > j \text{ integers.}$$

proof: On $[-1,1]$ we have.

$$T_n(x) = \cos(n \cos^{-1} x)$$

thus:

$$T_i(x)\, T_j(x) = \cos(i \cos^{-1} x) \cos(j \cos^{-1} x)$$

$$= \frac{1}{2}\left[\cos((i+j)\cos^{-1} x) + \cos((i-j)\cos^{-1} x) \right]$$

$$= \frac{1}{2}\left[T_{i+j}(x) + T_{i-j}(x) \right]$$

```
% Problem 1: B&F 8.1.6 a b c

% data points abscissa and ordinates
x = [0.2,      0.3,      0.6,     0.9,     1.1,    1.3,    1.4,    1.6]';
y = [0.050446, 0.098426, 0.33277, 0.72660, 1.0972, 1.5697, 1.8487, 2.5015]';

% best approximation first degree
A = [ ones(size(x)), x];
c1=(A'*A)\(A'*y);
fprintf('Best approximation is %g + %g x\n',c1);
fprintf('error ||b-A*x||_2^2 for degree 1 approx = %g\n',norm(A*c1-y)^2);

% best approximation second degree
A = [ ones(size(x)), x, x.^2];
c2=(A'*A)\(A'*y);
fprintf('Best approximation is %g + %g x + %g x^2 \n',c2);
fprintf('error ||b-A*x||_2^2 for degree 2 approx = %g\n',norm(A*c2-y)^2);

% best approximation third degree
A = [ ones(size(x)), x, x.^2, x.^3];
c3=(A'*A)\(A'*y);
fprintf('Best approximation is %g + %g x + %g x^2 + %g x^3\n',c3);
fprintf('error ||b-A*x||_2^2 for degree 3 approx = %g\n',norm(A*c3-y)^2);
```



```
>> prob1
Best approximation is -0.512457 + 1.66554 x
error ||b-A*x||_2^2 for degree 1 approx = 0.33559
Best approximation is 0.0851439 + -0.311403 x + 1.12942 x^2
error ||b-A*x||_2^2 for degree 2 approx = 0.00241991
Best approximation is -0.0184014 + 0.248386 x + 0.402932 x^2 + 0.266208 x^3
error ||b-A*x||_2^2 for degree 3 approx = 5.07467e-06
```