§ 2.2 Fixed point iteration

Recall Newton's method iteration:

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

This can be rewritten in the form:

\[ x_{n+1} = F(x_n) \quad (*) \]

where \( F(x) = x - \frac{f(x)}{f'(x)} \)

An iteration of the form (*) is called fixed point iteration, since if \( x_n \to x^* \) and \( F(x) \) is continuous, then \( x^* \) must be a fixed point of \( F(x) \), i.e., \( x^* = F(x^*) \).

This is easily seen from def on continuity:

\[ F(x^*) = F \left( \lim_{n \to \infty} x_n \right) = \lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} x_{n+1} = x^* \]

Fact.
Of course not all iterations of the form \((*)\) converge, an easy counterexample being:

\[ F(x) = 2x \rightarrow x_n = 2^n x_0 \text{ which diverges!} \]

The next theorem is very powerful and ubiquitous in applied math and gives a condition under which \(F\) exists and is unique, and iteration \((*)\) converges to that particular fixed point.

But first recall the def: \(f\) is Lipschitz continuous with Lipschitz constant \(L\):

\[ \forall x, y \quad |f(x) - f(y)| \leq L |x - y| \]

**Theorem:** (Contractive mapping theorem)

\[ \text{Let } C \text{ be a closed and bounded set of } \mathbb{R}. \]
\[ \text{If } F \text{ is Lipschitz continuous with Lipschitz constant } L < 1, \]
\[ \text{then } F \text{ has a unique fixed point } x^* \in C. \]
\[ \text{Moreover iteration } (*) \text{ converges to } x^* \text{ regardless of initial iterate } x_0 \in C. \]

**Proof:**

\[ |x_n - x_{n-1}| = |F(x_{n-1}) - F(x_{n-2})| \]
\[ \leq L |x_{n-1} - x_{n-2}| \]

Thus:

\[ |x_n - x_{n-1}| \leq L |x_{n-1} - x_{n-2}| \]
\[ \leq L^2 |x_{n-2} - x_{n-3}| \]
\[ \vdots \]
\[ \leq L^{n-1} |x_1 - x_0| \]
Now we can write an as a telescoping series:

\[ a_n = x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \]

\[ = x_0 + \sum_{k=1}^{n} (x_k - x_{k-1}) \]

\[ \{x_n\} \text{ converges } \iff \sum_{k=1}^{\infty} x_k - x_{k-1} \text{ converges.} \]

To show that the latter series converges it suffices to show that:

\[ \sum_{k=1}^{\infty} |x_k - x_{k-1}| \text{ converges.} \]

This is true since \[ |x_k - x_{k-1}| \leq L^{k-1} |x_1 - x_0| \]

which means series is dominated by a geometric series:

\[ \sum_{k=1}^{\infty} |x_k - x_{k-1}| \leq \sum_{k=1}^{\infty} L^{k-1} |x_1 - x_0| = \frac{1}{1-L} |x_1 - x_0|. \]

Now let \( x^* = \lim_{n \to \infty} a_n = x_0 + \lim_{n \to \infty} \sum_{k=1}^{n} x_k - x_{k-1} \)

by continuity of \( F \), we have \( x^* = F(x^*) \), so the limit is a fixed point.

What about uniqueness? Assume there are two fixed points \( x^* \) and \( y^* \), with \( x^* \neq y^* \) for contradiction. Then:

\[ |x^* - y^*| = |F(x^*) - F(y^*)| \leq L |x^* - y^*| \]

\[ < |x^* - y^*| \]

contradiction.

\[ \Rightarrow x^* = y^* \text{ and fixed point is unique.} \]
One other way of proving a contractive mapping theorem is to invoke Cauchy's criterion.

A Cauchy sequence \( \{x_n\} \) is a sequence s.t.
\[
\forall \varepsilon > 0 \exists N \quad \forall n, m \geq N \quad |x_n - x_m| < \varepsilon
\]

The real line \( \mathbb{R} \) is an example of a complete metric space where every Cauchy sequence converges inside the space \( \mathbb{R} \).

So to show convergence of F.P. iteration we can also use Cauchy criterion: \( (n \geq m \geq N_0) \)

\[
|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + x_{m+1} + x_{m+1} - x_m|
\]
\[
\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|
\]
\[
\leq L^{n-1} |x_1 - x_0| + L^{n-2} |x_2 - x_0| + \cdots + L^m |x_n - x_0|
\]
\[
= L^m |x_1 - x_0| (1 + L + L^2 + \cdots + L^{n-m-1})
\]
\[
= L^m |x_1 - x_0| \frac{1 - L^{n-m}}{1 - L}
\]
\[
\leq \frac{L^{N_0}}{1 - L} |x_1 - x_0| \rightarrow \text{can be made as small as possible provided } N_0 \text{ is large enough.}
Example: Show that $f_{n+3}$ is convergent:

$$
\begin{align*}
&\begin{cases}
x_0 = -15 \\
x_{n+1} = 3 - \frac{1}{2} |x_n|
\end{cases} \\
F(x) = 3 - \frac{1}{2} |x| is a contraction since: \\
\frac{1}{2} |x| - |y| &\leq \frac{1}{2} |x-y| \\
\Rightarrow F(x) &\rightarrow \text{unique fixed point of } F, \text{ which is } 2.
\end{align*}
$$

Error analysis

Consider the F. P. iteration:

$$
\begin{align*}
&\begin{cases}
x_0 = \text{given} \\
x_{n+1} = F(x_n)
\end{cases} \\
\text{with } x_n \rightarrow x* as n \rightarrow \infty.
\end{align*}
$$

Assuming $F \in C^1$; we can apply MVT

$$
x_{n+1} - x* = F(x_n) - F(x*) = F'(\xi_n)(x_n - x*),
$$

for some $\xi_n$ between $x_n$ and $x*$.

Thus having $|F'(\xi_n)| < 1$ guarantees convergence.

(You can see that this $\Rightarrow$ Lipschitz constant < 1)

Since close to fixed point $F'(\xi_n) \approx F'(x*)$ we could ask what happens in the extreme case where

$$
F'(x*) = 0.
$$

We shall see what happens in the more general case:

$$
F(k)(x*) = 0 \quad \text{for } 1 \leq k < q \quad (\#1)
$$

but

$$
F(q)(x*) \neq 0.
$$
We use Taylor's theorem again:

\[
\begin{align*}
    x_{n+1} - x^* &= F(x_n) - F(x^*) \\
    &= F(x^* + x_n - x^*) - F(x^*) \\
    &= F(x^*) + (x_n - x^*)F'(x^*) + \frac{1}{2}(x_n - x^*)^2 F''(\xi) + \ldots \\
    &- F(x^*) \\
    &= \sum_{k=1}^{q-1} \frac{(x_n - x^*)^k}{k!} F^{(k)}(x^*) + \frac{1}{q!} (x_n - x^*)^q F^{(q)}(\xi_n)
\end{align*}
\]

where \(\xi_n\) is between \(x_n\) and \(x^*\).

Under assumption (*) the first \(q-1\) derivatives of \(F\) vanish at \(x^*\), so:

\[
    x_{n+1} - x^* = \frac{1}{q!} (x_n - x^*)^q F^{(q)}(\xi_n)
\]

\[
| x_{n+1} - x^* | \leq \frac{|F^{(q)}(\xi_n)|}{q!} |x_n - x^*|^q
\]

bounded if \(F^{(q)}\) is continuous

\[
\leq M |x_n - x^*|^q
\]

\[
\Rightarrow x_n \to x^* \text{ with order of convergence } q.
\]

**Example:** We can show quadratic convergence of Newton's method using the above:

\[
F(x) = x - \frac{f(x)}{f'(x)}
\]

\[
F'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}
\]
Some fixed point of $F$ is a zero of $f$:

$$F'(x_*) = \frac{f(x_*) f''(x_*)}{[f'(x_*)]^2} = 0$$

$$F''(x) = \frac{(f')^2 (ff''' + f''f') - (2f''f')(ff'')}{(f')^4}$$

$$\Rightarrow F''(x_*) = \frac{f''(x_*)}{f'(x_*)} \neq 0 \quad (usually)$$

In general, the order of convergence of a fixed point iteration is the first integer $q$ for which $F^{(q)}(x_*) \neq 0$.

**Graphical interpretation**

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"smooth"

"staircase"
Newton's method and multiple roots of a function

**Def.** A root $x^* \neq 0$ of $f(x)$ is said to be a root of multiplicity $q$ of $f$ if we can write:

$$f(x) = (x - x^*)^q \cdot g(x)$$

with $\lim_{x \to x^*} g(x) \neq 0$.

One can easily identify such zeros by looking at derivatives:

**Thm.** $f \in \mathcal{C}^2[a,b]$ has a simple zero in $(a,b)$ iff $f(x^*) = 0$ but $f'(x^*) \neq 0$.

**Proof.** $\Rightarrow$: if $f$ has a simple zero at $x^*$, then:

$$f(x^*) = 0 \quad \text{and} \quad f'(x^*) \neq 0 \quad \text{with} \quad \lim_{x \to x^*} g(x) \neq 0.$$

But $f \in \mathcal{C}^2[a,b]$ so:

$$f'(x^*) = \lim_{x \to x^*} f'(x) = \lim_{x \to x^*} g(x) + (x - x^*)g'(x)$$

$$= \lim_{x \to x^*} g(x) \neq 0.$$

$\Leftarrow$: if $f(x^*) = 0$ but $f'(x^*) \neq 0$ we can use Taylor's theorem around $x^*$:

$$f(x) = f(x^*) + f'(x^*) (x - x^*)$$

$$= (x - x^*) f'(x^*)$$

where $x(x)$ is some point between $x$ and $x^*$. 
We obviously have: \( \lim_{x \to x^*} f(x) = x^* \).

and by continuity of \( f \):

\[ \lim_{x \to x^*} f'(f(x)) = f' \left( \lim_{x \to x^*} f(x) \right) = f'(x^*) \neq 0 \]

We then choose \( g(x) = f'(f(x)) \)

\[ \Rightarrow \quad f(x) = (x-x^*)g(x) \]

\[ \Rightarrow \quad x^* \text{ is a simple zero of } f. \]

One can easily extend this theorem (HW) to higher multiplicities.

Then the function \( f \in C^m[a,b] \) has a zero \( x^* \) of multiplicity \( q \) if and only if:

\[ 0 = f(x^*) = f'(x^*) = f''(x^*) = \cdots = f^{(q-1)}(x^*) \]

but \( 0 \neq f^{(q)}(x^*). \)

If \( x^* \) is not a simple zero of \( f(x) \) then Newton's method may converge but not quadratically.

Example: \( f(x) = e^x - x - 1 \), \( f(0) = e^0 - 0 - 1 = 0 \)

\[ f'(0) = e^0 - 1 = 0 \]

\[ f''(0) = e^0 \neq 0 \]

\( 0 \) is a zero of multiplicity 2 of \( f(x) \).