MATH 5610/6860
HOMEWORK #4, PROBLEM 6 HINT

In **HW4** Problem 6 (extra credit) you are asked to prove **Leibniz formula** for divided differences:

$$(fg)[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f[x_0, x_1, \ldots, x_k] g[x_k, x_{k+1}, \ldots, x_n].$$

The proof is longer and more challenging than I thought so here are the main steps.

i. Let $p(x)$ be the polynomial of degree $\leq n$ interpolating f at $x_0, x_1, \ldots, x_n$. Explain why it is true that

$$(fg)[x_0, x_1, \ldots, x_n] = (pg)[x_0, x_1, \ldots, x_n].$$

ii. Divided differences are linear in the sense that for any constants α, β and any functions u, v,

$$(\alpha u + \beta v)[x_0, \ldots, x_n] = \alpha u[x_0, \ldots, x_n] + \beta v[x_0, \ldots, x_n].$$

By writing $p(x)$ in Newton form with divided differences, show that

$$(fg)[x_0, \ldots, x_n] = \sum_{k=0}^{n} f[x_0, \ldots, x_k] \left(\left(\prod_{j=0}^{k-1} (x - x_j) \right) g(x) \right) [x_0, \ldots, x_n].$$

iii. Let k be an index between 0 and n. Show that

$$g[x_k, x_{k+1}, \ldots, x_n] = \left(\left(\prod_{j=0}^{k-1} (x - x_j) \right) g(x) \right) [x_0, \ldots, x_n].$$

Hint: Construct the polynomial interpolating the function of x $\left(\prod_{j=0}^{k-1} (x - x_j) \right) g(x)$ at the nodes $x_0, \ldots, x_n$ using the polynomial interpolating $g(x)$ at $x_k, x_{k+1}, \ldots, x_n$.

iv. Use the last two steps to conclude.