In HW4 Problem 6 (extra credit) you are asked to prove Leibniz formula for divided differences:

\[
(fg)[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f[x_0, x_1, \ldots, x_k]g[x_k, x_{k+1}, \ldots, x_n].
\]

The proof is longer and more challenging than I thought so here are the main steps.

i. Let \( p(x) \) be the polynomial of degree \( \leq n \) interpolating \( f \) at \( x_0, x_1, \ldots, x_n \).

Explain why it is true that

\[
(fg)[x_0, x_1, \ldots, x_n] = (pg)[x_0, x_1, \ldots, x_n].
\]

ii. Divided differences are linear in the sense that for any constants \( \alpha, \beta \) and any functions \( u, v \),

\[
(\alpha u + \beta v)[x_0, \ldots, x_n] = \alpha u[x_0, \ldots, x_n] + \beta v[x_0, \ldots, x_n].
\]

By writing \( p(x) \) in Newton form with divided differences, show that

\[
(fg)[x_0, \ldots, x_n] = \sum_{k=0}^{n} f[x_0, \ldots, x_k] \left( \prod_{j=0}^{k-1} (x - x_j) \right) g(x) [x_0, \ldots, x_n].
\]

iii. Let \( k \) be an index between 0 and \( n \). Show that

\[
g[x_k, x_{k+1}, \ldots, x_n] = \left( \prod_{j=0}^{k-1} (x - x_j) \right) g(x) [x_0, \ldots, x_n].
\]

**Hint:** Construct the polynomial interpolating the function of \( x \)

\[
\left( \prod_{j=0}^{k-1} (x - x_j) \right) g(x)
\]

at the nodes \( x_0, \ldots, x_n \) using the polynomial interpolating \( g(x) \) at \( x_k, x_{k+1}, \ldots, x_n \).

iv. Use the last two steps to conclude.