MATH 5610/6860

FINAL EXAM PRACTICE PROBLEMS

Problem 1:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{E_2 = E_2 - E_1 \\ E_3 = E_3 - E_1}]{} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

Note: do not multiply current equation by a constant otherwise L will not have diagonal of all ones.

$$L = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 0 & 1 \end{bmatrix}$$

Problem 2 (a) No pivoting

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow[\substack{E_2 = E_2 - 2E_1 \\ E_3 = E_3 - E_1}]{} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -5 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow{E_3 = E_3 - E_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -5 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 1 & 1 & 1 \end{bmatrix}$$

(b) With pivoting:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow[E_2 \leftrightarrow E_1]{} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow[\substack{E_2 = E_2 - \frac{1}{2}E_1 \\ E_3 = E_3 - \frac{1}{2}E_1}]{} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 5/2 \\ 0 & -1/2 & -1/2 \end{bmatrix}$$

$$\xrightarrow[E_3 = E_3 + E_2]{}$$

$$U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ \frac{1}{2} & -1 & 1 \end{bmatrix}$$

we have $LU = A([2\ 1\ 3], :)$

↑ row permutation. vector

Problem 3

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2.01 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 201 & -200 \\ -100 & 100 \end{bmatrix}$$

(a)

$$\|A\|_\infty = \max(3, 3.01) = 3.01; \quad \|A^{-1}\|_\infty = \max(200, 401) = 401$$

$$\Rightarrow \boxed{K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 3.01 \times 401 = 1207.01}$$

(b) $b = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \Rightarrow \|b\|_\infty = 4 \qquad \|b - \tilde{b}\|_\infty = 1$

$\tilde{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$x = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ $\quad \tilde{x} = A^{-1}\tilde{b} = \begin{bmatrix} -397 \\ 200 \end{bmatrix}$

$\| x - \tilde{x} \|_\infty = 401$

we have:

$$\frac{\|x-\tilde{x}\|_\infty}{\|x\|_\infty} = \frac{401}{4}$$

$$\frac{\|b-\tilde{b}\|_\infty}{\|b\|_\infty} = \frac{1}{4} \qquad \mathcal{K}_\infty(A) = 3.01 \times 401$$

Thus we do have:

$$\frac{401}{4} = \frac{\|x-\tilde{x}\|_\infty}{\|x\|_\infty} \leq \mathcal{K}(A)_\infty \frac{\|b-\tilde{b}\|_\infty}{\|b\|_\infty} = 3.01 \times \frac{401}{4}.$$

Problem 4: Let $D = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}$

The system we work with is: $\underbrace{D^{-1}A}_{\tilde{A}}\, x = \underbrace{D^{-1}b}_{\tilde{b}}$

Applying Richardson's method gives:

$$x^{(k+1)} = (I - \tilde{A})x^{(k)} + \tilde{b}$$

$$\Leftrightarrow \quad x^{(k+1)} = D^{-1}(D-A)x^{(k)} + D^{-1}b$$

$$\Leftrightarrow \quad D x^{(k+1)} = (D-A)x^{(k)} + b$$

$\equiv$ Jacobi method.

Problem 5:

$$P_3(x) = x^3 - \frac{3}{5}x$$

$$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

$$P_5(x) = xP_4(x) - \underbrace{\frac{(xP_4, P_4)}{(P_4, P_4)}}_{=0 \text{ because } x(P_4)^2 \text{ is odd}} P_4 - \frac{(xP_4, P_3)}{(P_3, P_3)} P_3$$

here we use notation $(f,g) = \int_{-1}^{1} f(x)g(x)\,dx$

Now: $$(xP_4, P_3) = \int_{-1}^{1}\left(x^5 - \frac{6}{7}x^3 + \frac{3}{35}x\right)\left(x^3 - \frac{3}{5}x\right)dx$$

$$= \frac{2}{9} - \frac{6}{35} - \frac{12}{49} + \frac{36}{175} + \frac{6}{175} - \frac{6}{175}$$

$$= \frac{128}{11025}$$

$$(P_3, P_3) = \int_{-1}^{1}\left(x^3 - \frac{3}{5}x\right)\left(x^3 - \frac{3}{5}x\right)dx = \frac{2}{7} - \frac{6}{25} - \frac{6}{25} + \frac{6}{25}$$

$$= \frac{8}{175}$$

$$\boxed{P_5(x) = xP_4(x) - \frac{16}{63}P_3} = x^5 + \left(-\frac{6}{7} - \frac{16}{63}\right)x^3 + \left(\frac{3}{35} + \frac{16}{63}\times\frac{3}{5}\right)x$$

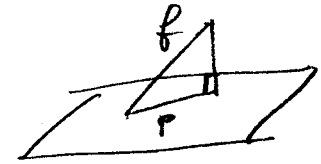
$$\boxed{= x^5 - \frac{10}{9}x^3 + \frac{5}{21}x}$$

Problem 6

The condition for optimality is that: $p(x) = a + bx$

$$(p - f, 1) = 0$$
$$(p - f, x) = 0$$

i.e.



This gives system:

$$\begin{bmatrix} (1,1) & (1,x) \\ (x,1) & (x,x) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (1, x^3) \\ (x, x^3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/5 \end{bmatrix} \Rightarrow \boxed{\begin{aligned} a &= -\tfrac{1}{5} \\ b &= \tfrac{9}{10} \end{aligned}}$$

Thus $\boxed{p(x) = -\frac{1}{5} + \frac{9}{10} x}$

Problem 7

$$\sum_{j=0}^{N-1} |p(x_j)|^2 = \sum_{j=0}^{N-1} \left[\sum_{k=0}^{N-1} c_k E_k(x_j) \right] \left[\sum_{k'=0}^{N-1} \overline{c_{k'}}\, \overline{E_{k'}}(x_j) \right]$$

$$= \sum_{k=0}^{N-1} \sum_{k'=0}^{N-1} c_k \overline{c_{k'}} \underbrace{\sum_{j=0}^{N-1} E_k(x_j) \overline{E_{k'}}(x_j)}_{N(E_k, E_{k'})_N = N \delta_{kk'}}$$

$$= N \sum_{k=0}^{N-1} |c_k|^2$$