MATH 5610/6860
FINAL EXAM PRACTICE PROBLEMS #2

1. Find the LU factorization (with L being a unit lower triangular matrix) of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2. Find the LU factorization (with L being a unit lower triangular matrix) of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

   (a) Without pivoting.
   (b) With scaled row pivoting. Clearly indicate which rows have been permuted.

3. (K&C 4.5.42) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2.01 \end{bmatrix}.$$

   (a) Compute the condition number $\kappa_\infty(\mathbf{A})$.
   (b) Verify that the bound for the relative error given by the condition number is satisfied with the right hand sides $\mathbf{b} = (4,4)^T$ and $\widetilde{\mathbf{b}} = (3,5)^T$.

4. (K&C 4.6.8) Show that if the $i-$th equation of the linear system $\mathbf{Ax} = \mathbf{b}$ is divided by a_{ii} and then Richardson's method is applied to solve the system, the result is the same as applying Jacobi's method.

5. Find the Legendre polynomial $p_5(x)$ from the two previous Legendre polynomials $p_3(x)$ and $p_4(x)$.

$$p_3(x) = x^3 - \frac{3}{5}x$$
$$p_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$
$$p_5(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}.$$

6. Find the polynomial of the form $p(x) = ax + b$ that best approximates $f(x) = x^3$ in $[0,1]$, where the norm is induced by the product

$$(f,g) = \int_0^1 f(x)g(x)dx.$$

7. Let $x_j = 2\pi j/N$ and $E_j(x) = \exp[2i\pi jx/N]$, $j = 0, \dots, N-1$. Recall the pseudo-inner product

$$(f, g)_N = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j)\overline{g}(x_j),$$

and its fundamental property

$$(E_n, E_m) = \begin{cases} 1 & \text{if } n - m \text{ is divisible by } N \\ 0 & \text{otherwise.} \end{cases}$$

Consider the trigonometric polynomial

$$p = \sum_{k=0}^{N-1} c_k E_k.$$

Show the discrete Parseval's identity

$$\sum_{j=0}^{N-1} |p(x_j)|^2 = N \sum_{j=0}^{N-1} |c_k|^2.$$

Hint: For $j = 0, \dots, N-1$ we have,

$$|p(x_j)|^2 = \left(\sum_{k=0}^{N-1} c_k E_k(x_j) \right) \left(\sum_{k'=0}^{N-1} c_{k'} \overline{E}_{k'}(x_j) \right).$$