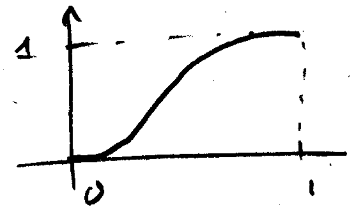
MATH 5610/6860 practice final problem
#1

problem 1 :

$S(0)=0 \qquad S(1)=1$

$S'(0)=0 \qquad S'(1)=0$



Use ansatz

$S(x)= ax^3+bx^2+cx+d$

$S'(x)= 3ax^2+2bx+c$

$S(0)=d=0$

$S'(0)=c=0$

$$\left.\begin{array}{l} S(1)=a+b=1 \\ S'(1)=3a+2b=0 \end{array}\right\} \Rightarrow \begin{array}{l} a=-2 \\ b=3 \end{array}$$

$$\boxed{S(x)=-2x^3+3x^2}$$

note: This is a clamped cubic spline.
The free cubic spline passing through two nodes is simply a line.

problem 2 (K & C 7.1.15)

We have:

$$(-1)\times \quad f'(x) = \underbrace{\frac{1}{2h}\left[f(x+h)-f(x-h)\right]}_{R(h)} - \frac{h^2}{6}f'''(x) - \frac{h^4}{120}f^{(5)}(x) \cdots$$

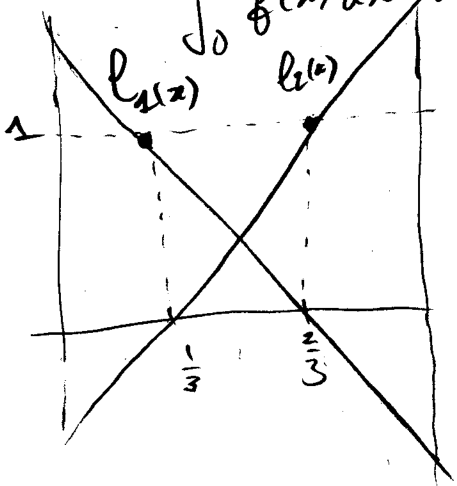
$$(4)\times \quad f'(x) = R(h/2) - \frac{h^2}{24}f'''(x) - \frac{h^4}{16\times 120}f^{(5)}(x) \cdots$$

$$3f'(x) = 4R(h/2) - R(h) + \frac{3}{4\times 120}h^4 f^{(5)}(x) \cdots$$

$$\boxed{f'(x) = \frac{4R(h/2) - R(h)}{3} + \frac{1}{480} h^4 f^{(5)}(x) + \cdots}$$

Problem 3
(Kec 7.2.10)

$$\int_0^1 f(x)\,dx \approx A f\left(\tfrac{1}{3}\right) + B f\left(\tfrac{2}{3}\right)$$



$$\int_0^1 l_1(x)\,dx = A = \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} = \frac{1}{2}$$

By symmetry:

$$\int_0^1 l_2(x)\,dx = B = \frac{1}{2}$$

$$\Rightarrow \int_0^1 f(x)\,dx \approx \frac{1}{2} f\left(\tfrac{1}{3}\right) + \frac{1}{2} f\left(\tfrac{2}{3}\right)$$

To use this formula on any interval $[a, b]$ we simply do a change of variables:

$$\int_a^b f(x)\,dx = \int_0^1 f((b-a)\tilde{x}+a)\,(b-a)\,d\tilde{x}$$

$$\tilde{x} = \frac{x-a}{b-a}$$

$$x = \tilde{x}(b-a) + a$$

$$\approx \frac{b-a}{2}\left[f\left(a + \frac{b-a}{3}\right) + f\left(a + \frac{2(b-a)}{3}\right)\right]$$

Problem 4

$$I \equiv \int_u^v f(x)\,dx = \int_u^w f(x)\,dx + \int_w^v f(x)\,dx$$
$$= T(u,w) - \frac{1}{2}(w-u)^3 f''(\xi_1)$$
$$+ T(w,v) - \frac{1}{2}(v-w)^3 f''(\xi_2)$$

To simplify expressions let $h = v-u \Rightarrow w-u = v-w = \frac{h}{2}$

Thus:

$$I = T(u,w) + T(w,v) - \frac{1}{2\times 8} h^3 \left[f''(\xi_1) + f''(\xi_2) \right]$$

Since $f''(\xi)$ is continuous by intermediate value theorem:

$$\exists\, \tilde{\xi} \in (u,v) \text{ s.t. } f''(\tilde{\xi}) = \frac{1}{2}\left[f''(\xi_1) + f''(\xi_2) \right]$$

Thus:

$$I = T(u,w) + T(w,v) - \frac{h^3}{8} f''(\tilde{\xi}) \Rightarrow \boxed{C = -\frac{1}{8}}$$

We combine the two equations so as to cancel out the $f''(\xi) \approx f''(\tilde{\xi})$ term.

$$\left(-\frac{1}{3}\right)\times \quad I = T(u,v) - \frac{1}{2} h^3 f''(\xi)$$
$$\left(\frac{4}{3}\right)\times \quad I = T(u,w) + T(w,v) - \frac{1}{8} h^3 f''(\tilde{\xi})$$

$$I \approx \frac{4}{3}\left(T(u,w) + T(w,v)\right) - T(u,v)$$
$$= T(u,w) + T(w,v) + \underbrace{\frac{1}{3}\left(T(u,w) + T(w,v) - T(u,v)\right)}_{\approx -\frac{1}{8} h^3 f''(\tilde{\xi})}$$