MATH 5610/6860 FINAL
STUDY SHEET

This exam covers §1.1 – 1.3, §2.1 –2.6, §3.1 – 3.2, 3.4, §4.1 – 4.7, §6.1–6.6, §7.1 – 7.5, §8.1 – 8.3, 8.5, 8.6 in the class textbook. We did not cover the following, so it is **not included in the exam**:

(1) Neville's method (in §3.1)
(2) Newton Forward and Backward divided differences (discussion starting after Theorem 3.6 in §3.2 and through the end of §3.2)
(3) Clamped cubic splines

A more detailed list of what you should know how to do is as follows

Chapter 1:

- preliminaries: limit, continuity, Rolle's theorem, extreme value theorem, intermediate value theorem, mean value theorem (MVT), MVT for integrals, Taylor's theorem (Lagrange remainder and integral remainder)
- order of convergence, little-oh and big-Oh notation.
- Floating point arithmetic: you should know how to do HW 1 problem 8, machine epsilon and its meaning.

Chapter 2:

- Bisection method: pseudocode, linear convergence
- Newton's method: pseudocode, graphical interpretation, quadratic convergence
- Secant method: pseudocode, graphical interpretation, super-linear convergence
- Fixed point iteration: pseudocode, graphical interpretation, contractive mapping theorem. What is a contraction? Condition on the derivative for a function to be a contraction. When can fixed point iteration give higher order of convergence? Use this principle to show Newton's method converges quadratically.
- How to modify Newton's method for multiple roots?
- Aitken's acceleration (just principle, not the formula!), Steffensen's method (pseudocode)
- Müller's method (pseudocode, graphical interpretation, don't learn the formulas!!)

Chapter 3:

- Preliminaries on polynomials: remainder theorem, factor theorem, show that two polynomials are identical by comparing at a finite (how many?) number of distinct points.
- Horner's algorithm: evaluate a polynomial and derivatives, deflation, Taylor's expansion. (you should know how to do these by hand, and know pseudocode at least for the basic version of Horner's algorithm)
- Interpolation: existence and uniqueness of interpolating polynomial, Newton form of interpolation polynomial, Lagrange form. You should know how to get coefficient directly and with divided differences. Interpolation error theorem in "Lagrange form" and with divided differences.
- Divided differences table to compute interpolating polynomial. Definition of divided differences. Recursive formula for divided differences.
- For cubic splines you should know what their definition and the continuity conditions. Please do not memorize the spline formulas! You should be able (with enough guidance) to equation for the segment of a spline.

Chapter 4:

- The key formulas to know here are the centered and one sided finite differences and their error terms. You should know how to obtain such formulas from the Taylor expansion and the generalized mean value theorem trick.
- You should also know that in general $n-$point differentiation formulas can be obtained by writing the Lagrange interpolating polynomial, differentiating and evaluating at one of the nodes.
- For Richardson's extrapolation the key is to remember the trick of combining two approximations with known form with two parameters (usually h and $h/2$) in order to cancel out the leading order term in the error. A good problem would be to find such a combination in a particular case just as is done in pp180-181
- For numerical integration you need to know the different methods to obtain quadrature formulas:

- Form Lagrange interpolation polynomial and integrate. This gets quickly long and tedious.
- Undetermined coefficient method: form a linear system imposing the condition that the quadrature be *exact* for $p \in \Pi_n$, where Π_n is the set of polynomials of degree n or less.

- Know midpoint, trapezoidal and Simpson rule error terms.
- What happens in general when you take an integration rule and make it composite (loose one order of h).
- For Romberg integration there is nothing to memorize, only that the idea is similar to Richardson's extrapolation and that you could be asked to find the proper linear combination that cancels out the leading order term in the error.
- For adaptive quadrature methods you should know the principle of the method. The only possible thing I could ask is to obtain the error estimation that predicts whether Simpson's rule (or trapezoidal?) approximation to the integral is within a certain tolerance.
- For Gaussian quadrature the key idea is that we allow the weights and nodes to be adjusted. For n nodes this gives $2n$ degrees of freedom so me can expect the formulas we obtain to be exact for polynomials of degree $2n - 1$. There is no need to remember the proofs we made in class or the expression for the orthogonal polynomials. It is important however to remember the Gram-Schmidt process to get the orthogonal polynomials and to be able to operate with them sufficiently well to verify facts in the proof of Theorem 4.7. What is a monic polynomial?

Chapter 6:

- Know how to do Gaussian elimination on a simple matrix (say up to 4×4). Partial pivoting. I could give an example similar to the ones we saw in class where things break down for Gaussian elimination but where the problems go away with pivoting.
- You do not need to remember the algorithm for the LU factorization, but you should know what the factorization means, what happens in the symmetric positive definite case (Cholesky). How much does it cost to compute L and U? What is the work involved in the forward and backward substitution? Can you write the algorithm for forward and/or backward substitution in pseudocode?
- What is a symmetric positive definite matrix?

Chapter 7:

- Neumann series, condition for convergence, spectral radius, induced matrix norm. Iterative refinement.
- Jacobi, Gauss-Seidel. Advantages, disadvantages and pseudocode.
- Condition number error bound (7.19) and interpretation. We did not see Theorem 7.29 and it is not included in the exam.
- Conjugate gradient: do not memorize algorithm. You should be able to explain what are the advantages and disadvantages of this method with respect to direct methods.

Chapter 8:

- Discrete least squares: normal equations and interpretation in terms of best approximation result by the element of a subspace (residual has to be orthogonal to subspace)
- Orthogonal polynomials and the simplification in Gram Schmidt procedure that gives a three term recurrence. Again: do not memorize these polynomials, only their properties.
- Chebyshev polynomials: If you know (8.8) the roots and number of sign changes follow. These polynomials are orthogonal with respect to what weight? Optimality property in Theorem 8.10 and application to "best" choice of interpolation nodes.
- Trigonometric interpolation and principle behind FFT (do not memorize algorithm). However I expect you to know what the FFT computes.

General:

- Know how to compute the inverse, determinant, characteristic polynomial, eigenvalues, eigenvectors of a 2×2 matrix or a simple 3×3 matrix.
- Know how to find the roots of a quadratic.
- Norms. Induced matrix norms ($\ell_1, \ell_2, \ell_\infty$, there is an easy mnemonic for these), spectral radius of simple matrices (2×2 or 3×3).