

Math 3150-4 – Final – review sheet

- §1.1-1.2 (preliminaries): What is a partial differential equation? Is a differential equation linear or non-linear? Homogeneous or non-homogeneous? What is the order of a differential equation?
- §2.1 (periodic functions) you should be able to sketch them, find their period, tell whether they are continuous, piecewise continuous and/or piecewise smooth and know how to integrate periodic functions. You should know at least the cosine and sine addition formulae and how to use them to derive the orthogonality relations of p21-22 (class notes: p18-20).
- §2.2 (Fourier series of a 2π -periodic function): What is the Fourier series of a 2π -periodic function? What are its Fourier coefficients? How can the Fourier coefficients be expressed with the orthogonality relations (i.e. formulas in p27 of book or p20-21 in class notes)? How can we express the integrals involved over some other period? What happens at the discontinuities of a piecewise smooth function?
- §2.3 (Fourier series of a function with arbitrary period). What is an orthogonal family of $2p$ -periodic functions? How do the orthogonality relations over $[-\pi, \pi]$ translate to functions on $[-p, p]$? What is the Fourier series of a $2p$ -periodic function? How to find the coefficients? What simplifications occur if the function is even or odd? What is the Fourier series of a sum of two functions?
- §2.4 (Half-range expansions) What happens if we extend a function from $[0, p]$ to a $2p$ -periodic odd (or even) function? What are the sine and cosine series of a function defined on $[0, p]$? How can we find the Fourier coefficients?
- §3.3 (1DWEQ with separation of variables) Know how to solve the 1D wave equation describing the vibrations of a string that is pinned down at its endpoints ($x = 0$ and $x = L$) (p119 in book or p38 in notes), with given initial position and velocity. How do the sine series coefficients of the initial position and initial velocity show up in the actual solution? What are the fundamental solutions of the 1DWEQ we obtained through separation of variables?
- §3.4 (D'Alembert's method) You should know the form of D'Alembert's solution to the 1D Wave Equation in the general case (both initial condition and initial velocity are given). You also should be able to verify that solutions to the wave equation given by separation of variables can be written in D'Alembert's form (see Example 3.4.1). Characteristics and Interval of dependence are not included.
- §3.5 (1D Heat equation) You should know how to solve the 1D Heat equation with homogeneous Dirichlet boundary conditions with separation of

variables. When the boundary conditions are Dirichlet non-homogeneous, the solution can be found by first finding the steady state heat distribution and subtracting it from the original problem to get a homogeneous one. Can you compare the rate of decay of two temperature distributions (say when they are made of pure modes, which decays faster, higher or lower frequencies?)

- §3.6 (1D Heat equation with other boundary conditions) You should be familiar with the case of homogeneous Neumann boundary conditions at both ends (Example 3.6.1). The key is to use cosine series instead of sine series, and the reason is that cosines satisfy the boundary conditions and sines do not! The case of one radiating end (Example 3.6.2) will not be included in the exam. However you are expected to know how to do the following mixed boundary cases for a bar $[0, L]$
 - Insulated at $x = 0$ and ice bath at $x = L$
 - Ice bath at $x = 0$ and insulated at $x = L$ (see Problem 3.6.5)
- §3.7 (2D WEQ and HEQ in rectangular domains with homogeneous Dirichlet boundary conditions) You should know how to do example 3.7.1 (and also the non-zero velocity case too). Since the double sine series coefficients calculation can be lengthy, it is very likely that either the coefficients will be given or the initial position or velocity of the membrane are given already in double sine series form (as in problem 3.7.1 for example). The solution to the 2D heat equation on a rectangular domain uses exactly the same double sine series tool and you should also be familiar with it (see example 3.7.2).
- §3.8 (2D Laplace equation in rectangular coordinates) Here the trick is to reduce the problem with boundary conditions on all four sides of the rectangle to 4 problems, each with a non-zero boundary condition on one side only. You only need to know how to do one of these problems (say the one in Example 3.8.1). The other 3 problems can be obtained by symmetries.
- §4.1: Here you should know what are polar coordinates and what is the Laplacian in polar coordinates. The derivation is not included in the exam (it is too long). You may be asked to compute $\partial r/\partial x$, $\partial r/\partial y$, $\partial \theta/\partial x$, $\partial \theta/\partial y$ using the chain rule or a problem similar to problem 4.1.1 (i.e. check whether an expression which is much simpler in polar coordinates is a solution to Laplace equation).
- §4.2 (Vibrations of circular membrane in the case where the initial conditions are independent of θ) You should know Theorem 4.2.1 and example 4.2.1 (without the numerical calculations of the actual coefficients of course). Since the calculations of the coefficients in a Bessel series expansion are hard to do by hand, you will likely be given the initial condition

and/or initial velocity in Bessel series expansion form already or be provided with Bessel function identities such as those used in Example 4.2.2. You may also be asked to consider the heat equation on a disk with radial symmetry (i.e. with initial condition independent of θ), see Problem 4.3.10.

- §4.3 (Vibrations of a circular membrane in the general case) The key here is that the product solutions (equations (5) and (6) in this chapter) depend on the angle too. The dependence on the radius involves higher order Bessel functions of the first kind (J_m) for which there are orthogonality relations (which will be provided in a formula sheet, they are given in p252 of your book). The dependence on angle is by the usual $\cos m\theta$ and $\sin m\theta$ because the membrane position should be 2π periodic in angle. Once you know the form of the product solutions and the orthogonality relations it should be doable to get the expressions (12)–(14) in your book (and you may be asked to do this and to evaluate the integrals in a case where it is easy to do by hand).
- §4.4 (Laplace equation in circular regions) You will be reminded at least of the equation that is satisfied by the radial part of the solution and its solution (this is Euler's equation, equation (3) in this chapter). You should know how to solve Examples 1 and 4. The boundary conditions may be varied (see Homework 9 solutions, which will be posted on the last day of classes). *Isotherms were not covered so they are not included in the exam.*
- §4.5 (Laplace equation in a cylinder) We considered the following cases:
 - a non-zero radially symmetric Dirichlet boundary condition on the top and zero elsewhere. This is essentially like the Laplace equation on a rectangular domain. The difference here is that z dependence is a \sinh and the radial dependence is a Bessel function J_0 (instead of a sine in the rectangular case).
 - a non-zero Dirichlet boundary condition depending only on the height z on the lateral side and zero on the top and bottom. Here the z dependence is a \sin and the radial dependence involves *modified* Bessel functions which are purely decaying (or increasing) instead of being oscillatory. Think of them as being to the usual Bessel functions the same as hyperbolic trigonometric functions are to the regular trigonometric functions. (this analogy goes even further: the modified Bessel functions can be obtained by looking at what happens to the usual Bessel functions on the imaginary axis).

Since we did not do examples in class, any exam problem on this topic will be heavily guided (i.e. you do not need to memorize formulas in this chapter or write them on your cheat sheet).