Math 3150 HW9 Solutions

4.4.10 (a)

$$\begin{cases} \Delta u = 0 \\ u(1,\theta) = 1 + \sin 2\theta \end{cases}$$

The general sol is:

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

With BC:

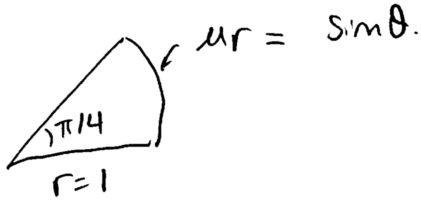
$$\underset{1+\sin 2\theta}{\overset{\shortparallel}{u(1,\theta)}} = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

$\Rightarrow$ $1 + \sin 2\theta$ is already in Fourier series form:

$$a_0 = 1$$
$$a_n = 0, \quad n > 0$$
$$b_n = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \boxed{u(r,\theta) = 1 + r^2 \sin 2\theta}$$

4.4.13



We proceed as in example 4.4.4.
The general form of solutions is still:

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n\, r^{\frac{n\pi}{\alpha}} \sin \frac{n\pi}{\alpha}\theta \quad , \alpha = \frac{\pi}{4}$$

$$= \sum_{n=1}^{\infty} b_n\, r^{4n} \sin(4n\theta)$$

$$u_r(r,\theta) = \sum_{n=1}^{\infty} (4n) b_n\, r^{4n-1} \sin(4n\theta)$$

$$u_r(1,\theta) = \sin\theta = \sum_{n=1}^{\infty} 4n\, b_n \sin(4n\theta)$$

$$\Rightarrow 4n\, b_n = \frac{(\sin\theta, \sin(4n\theta))}{(\sin(4n\theta), \sin(4n\theta))}$$

where $(u,v) = \int_0^{\pi/4} u(\theta)v(\theta)\, d\theta$

so:

$$(\sin(4n\theta), \sin(4n\theta)) = \int_0^{\pi/4} (\sin 4n\theta)^2\, d\theta = \frac{\pi}{8}$$

$$(\sin\theta, \sin(4n\theta)) = \int_0^{\pi/4} \sin\theta \sin 4n\theta\, d\theta$$

$$\overset{\text{add form.}}{=} \frac{1}{2}\int_0^{\pi/4} (\cos(4n-1)\theta - \cos(4n+1)\theta)\, d\theta$$

$$= \frac{1}{2}\left(\frac{\sin(4n-1)\theta}{4n-1} - \frac{\sin(4n+1)\theta}{4n+1} \right)\Bigg|_0^{\pi/4}$$

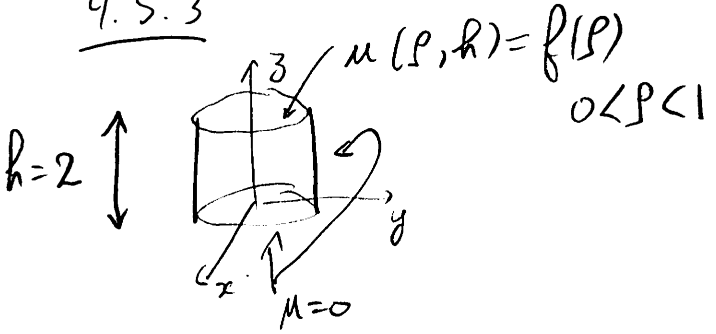
$$= -\frac{1}{2}(-1)^n \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{4n-1} + \frac{1}{4n+1}\right)$$

$$\Rightarrow (\sin\theta, \sin 4n\theta) = -\frac{4n(-1)^n(\sqrt{2}/2)}{(4n-1)(4n+1)}$$

$$\Rightarrow \quad b_n = \frac{(\sin\theta, \sin 4n\theta)}{(\sin 4n\theta, \sin 4n\theta)} = -\frac{8}{\pi}\frac{(-1)^n(\sqrt{2}/2)}{(4n-1)(4n+1)}$$

$$\Rightarrow \boxed{u(r,\theta) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}\sqrt{2}}{\pi(4n-1)(4n+1)} r^{4n} \sin(4n\theta)}$$

4.5.3



Solve $\Delta u = 0$ with
$$f(\rho) = \begin{cases} 100 & \text{if } 0 < \rho < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < \rho < 1 \end{cases}$$

$$\Rightarrow \begin{cases} \Delta u = 0 \\ u(\rho, 0) = 0 \\ u(1, z) = 0 \\ u(\rho, 2) = f(\rho) \end{cases}$$

Sol is of the form:
$$u(\rho, z) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \rho)\sinh(\lambda_n z)$$
where $\lambda_n = \frac{\alpha_n}{1}$ = n-th zero of 0-th order Bessel fun. $J_0(\cdot)$

and
$$A_n = \frac{2}{\sinh(2\alpha_n) J_1^2(\alpha_n)} \int_0^1 f(\rho) J_0(\alpha_n \rho)\rho\, d\rho$$
$$= (\quad '' \quad) \, 100 \int_0^{\frac{1}{2}} J_0(\alpha_n \rho)\rho\, d\rho$$

4

$$A_n = \left('' \right) 100 \int_0^{\alpha_n/2} J_0(x) \frac{x\,dx}{\alpha_n^2}$$

COV.

$x = \alpha_n \rho$

$dx = \alpha_n d\rho$

$$= \frac{200}{\sinh(2\alpha_n) J_1^2(\alpha_n) \alpha_n^2} \; x J_1(x) \Big|_0^{\alpha_n/2}$$

$$= \boxed{\frac{100 \, J_1(\alpha_n/2)}{\sinh(2\alpha_n) J_1^2(\alpha_n) \alpha_n}}$$