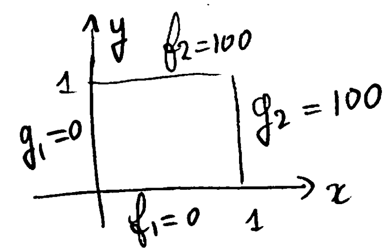
# HW 8 Solutions

**3.8.2**




We need to solve $\Delta u = 0$ w/ B.C. as given on the right.

The solution is:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sinh(n\pi y) + \sum_{n=1}^{\infty} D_n \sinh(n\pi x)\sin(n\pi y)$$

where:

$$B_n = \frac{2}{\sinh(n\pi)}\int_0^1 100 \sin n\pi x \, dx = \frac{200}{\sinh(n\pi)} \left. \frac{-\cos n\pi x}{n\pi}\right|_0^1 = \boxed{\frac{200(1-(-1)^n)}{n\pi \sinh(n\pi)}}$$

and

$$\boxed{D_n = \frac{2}{\sinh n\pi}\int_0^1 100 \sin n\pi y \, dy = B_n}$$

**4.1.2** $u(x,y) = \tan^{-1}\left(\frac{y}{x}\right) = \theta$

$$\Rightarrow \boxed{\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0}$$

**4.1.3** $u(x,y) = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

$$\Rightarrow u_r = -\frac{1}{r^2}$$

$$u_{rr} = +\frac{2}{r^3}$$

$$\Rightarrow \boxed{\Delta u = \frac{2}{r^3} - \frac{1}{r^3} = \frac{1}{r^3} \neq 0}$$

4.22 We want to solve the problem:

$$\begin{cases} u_{tt} = 10^2\left(u_{rr} + \frac{1}{r}u_r\right) \\ u(r,0) = 1-r^2 \\ u_t(r,0) = 1 \\ u(1,t) = 0 \end{cases}$$

we need to compute $A_n$ and $B_n$ as in Thm 1 of this chapter.

For $A_n$ we use method of example 2:

$$A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 (1-r^2)\, J_0(\alpha_n r)\, r\, dr$$

$$\overset{s=\alpha_n r}{=} \frac{2}{\alpha_n^4 J_1^2(\alpha_n)} \int_0^{\alpha_n} (\alpha_n^2 - s^2)\, J_0(s)\, s\, ds$$

$$\overset{IBP}{=} \frac{2}{\alpha_n^4 J_1^2(\alpha_n)} \left[ (\alpha_n^2 - s^2) J_1(s)\, s \Big|_0^{\alpha_n} + 2\int_0^{\alpha_n} J_1(s) s^2\, ds \right]$$

$$= \frac{4}{\alpha_n^4 J_1^2(\alpha_n)} \int_0^{\alpha_n} J_1(s) s^2\, ds = \frac{4}{\alpha_n^4 J_1^2(\alpha_n)} \alpha_n^2 J_2(s)\Big|_0^{\alpha_n}$$

$$= \boxed{\frac{4 J_2(\alpha_n)}{\alpha_n^2 J_1^2(\alpha_n)}}$$

with more simplification: (see example 4.2.2)

$$\boxed{A_n = \frac{8}{\alpha_n^3 J_1(\alpha_n)}}$$

For the $B_n$ we have:

$$\boxed{\frac{\alpha_n c}{1} B_n} = \frac{2}{J_1^2(\alpha_n)} \int_0^1 J_0(\alpha_n r)\, r\, dr$$

$$= \frac{2}{\alpha_n^2 J_1^2(\alpha_n)} \int_0^{\alpha_n} J_0(s)\, s\, ds$$

$$= \frac{2}{\alpha_n^2 J_1^2(\alpha_n)} s J_1(s) \Big|_0^{\alpha_n} = \boxed{\frac{2}{\alpha_n J_1(\alpha_n)}}$$

with $c^2 = 10^2$ we get:

$$B_n = \frac{1}{5\alpha_n^2 J_1(\alpha_n)}$$

$$\Rightarrow \boxed{u(r,t) = \sum_{n=1}^{\infty} J_0(\alpha_n r) \left[ \frac{8}{\alpha_n^3 J_1(\alpha_n)} \cos(10\alpha_n t) + \frac{1}{5\alpha_n^2 J_1(\alpha_n)} \sin(10\alpha_n t) \right]}$$

4.2.4 We would like to solve the problem.

$$\begin{cases} u_{tt} = u_{rr} + \frac{1}{r} u_r \\ u(r,0) = 0 \\ u_t(r,0) = J_0(\alpha_3 r) \\ u(1,t) = 0 \end{cases}$$

Since $u(r,0) = 0 \Rightarrow \boxed{A_n = 0}$

and:

$$\alpha_n B_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 J_0(\alpha_n r) J_0(\alpha_3 r)\, r\, dr$$

$$= \begin{cases} 0 & \text{if } n \neq 3 \\ 1 & \text{if } n = 3 \end{cases}$$

here we used $\perp$ relations in §4.8 p 252, thm 1.

$$\Rightarrow \boxed{u(r,t) = \frac{J_0(\alpha_3 r)}{\alpha_3} \sin(\alpha_3 t)}$$