MATH 3150 – HW6 solutions

3.4.1

$f(x) = \sin(\pi x)$, $g(x) = 0$, $c = \frac{1}{\pi}$, $x \in [0,1]$

f is already odd and 2 periodic, so no need to do odd extension

$$\boxed{u(x,t) = \frac{1}{2}\left[\sin \pi\left(x+\frac{t}{\pi}\right) + \sin\left(\pi\left(x-\frac{t}{\pi}\right)\right)\right] = \frac{1}{2}\left[\sin(\pi x + t) + \sin(\pi x - t)\right]}$$

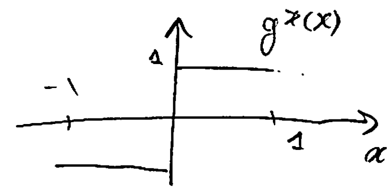
↑ D'Alembert's solution.

3.4.4

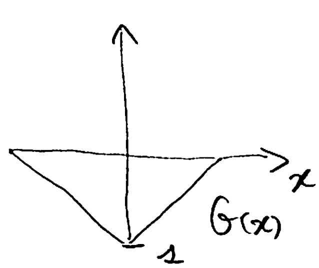
$f(x) = 0$, $g(x) = 1$, $c = 1$

For $x \in [-1,1]$ we have:

$$g^*(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ -1 & \text{if } -1 \le x < 0 \end{cases}$$



$$G(x) = \int_{-1}^{x} g^*(s)\,ds = \begin{cases} x-1 & \text{if } 0 \le x \le 1 \\ -x-1 & \text{if } -1 \le x < 0 \end{cases}$$



$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left(G(x+t) - G(x-t)\right)}$$

⚠ integration constant (i.e. antiderivative) does not matter.

3.4.9 (Extra Credit)

$$u(x, t+2\pi) = \frac{1}{2}\left[\sin(\pi x + t + 2\pi) + \sin(\pi x - t - 2\pi)\right]$$
$$= \frac{1}{2}\left[\sin(\pi x + t) + \sin(\pi x - t)\right]$$
$$= u(x, t)$$

$\Rightarrow$ u is 2π-periodic

$\Rightarrow$ string will pass though its initial pos. every 2π (units of time)

3.5.4

Solve

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = u(\pi, t) = 0 \\ u(x,0) = f(x) = \begin{cases} 100 & \text{if } 0 < x < \pi/2 \\ 0 & \text{if } \frac{\pi}{2} < x < \pi \end{cases} \end{cases}$$

We have:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx}$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$
$$= \frac{2}{\pi}\int_0^{\pi/2} 100 \sin nx \, dx = -\frac{200}{\pi} \frac{\cos nx}{n}\Big|_0^{\pi/2}$$
$$= \boxed{\frac{200}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)}$$

3.5.14 Solve

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 100 \\ u(x,0) = f(x) \quad (\text{as in } 3.5.4) \end{cases}$$

Steady state temp. distribution: $\boxed{s(x) = \frac{100}{\pi} x}$

(line s.t. $s(0)=0$, $s(\pi)=100$)

$\Rightarrow$ $v(x,t) = u(x,t) - s(x)$ solves

$$\begin{cases} v_t = v_{xx} \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = f(x) - s(x) \end{cases}$$

$$\Rightarrow v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx, \text{ where}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (f(x) - s(x)) \sin n\pi x \, dx = \overbrace{\frac{2}{\pi}\int_0^{\pi} f(x) \sin n\pi x \, dx}^{\text{from } 3.5.4} - \frac{2}{\pi}\int_0^{\pi} \frac{100}{\pi} x \sin n\pi x \, dx$$

$$= \frac{200}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) - \frac{200}{\pi^2}\left[\cancel{\frac{1}{n^2}\sin nx}^{\,0} - \frac{x}{n}\cos nx\right]_0^{\pi}$$

$$= \frac{200}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) + \frac{200}{\pi^2}\,\frac{(-1)^n \pi}{n}$$

$$= \frac{200}{n\pi}\left(1 + (-1)^n - \cos\frac{n\pi}{2}\right)$$

$$= \frac{200}{n\pi}\begin{cases} 1 & \text{if } n \equiv 0 \bmod 4 \\ 0 & n \equiv 1 \bmod 4 \\ 3 & n \equiv 2 \bmod 4 \\ 0 & n \equiv 3 \bmod 4 \end{cases}$$

	$\cos\frac{n\pi}{2}$	$(-1)^n$
0	1	1
1	0	-1
2	-1	1
3	0	-1

$$\Rightarrow \boxed{u(x,t) = V(x,t) + V(x) = \frac{100}{\pi}x + \sum_{n=1}^{\infty} \frac{200}{n\pi}\left(1 + (-1)^n - \cos\frac{n\pi}{2}\right) e^{-n^2 t} \sin nx.}$$

3.5.9

(a) $s(x) = 100x$ since $s(0)=0$

$s(1) = 100$

(b) $s(x) = 100$ since $s(0) = 100$

$s(1) = 100$

3.6.21

Solve
$$\begin{cases} u_t = u_{xx} \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = \cos \pi x \end{cases}$$

The initial temp distrib is already given as its cosine series.

$\Rightarrow$ $a_1 = 1$, $a_n = 0$ for $n \neq 1$.

$$\Rightarrow \boxed{u(x,t) = \cos \pi x \, \exp\left[-(n\pi)^2 t\right]}$$