MATH 3150 HW 3 Solutions

2.1.9

(a) $f(x+T)\,g(x+T) = f(x)\,g(x)$ $\Rightarrow$ prod of T-per fun is T-per

$$\frac{f(x+T)}{g(x+T)} = \frac{f(x)}{g(x)} \Rightarrow \text{quotient of T-per fun is T-per}$$

(b) Let f be a T-periodic function.

then: $f\left(\frac{x+aT}{a}\right) = f\left(\frac{x}{a}+T\right) = f\left(\frac{x}{a}\right)$

$\Rightarrow$ $f\left(\frac{x}{a}\right)$ is aT periodic.

(c) Let f be a T periodic function and g some other fun.

then:
$$g(f(x+T)) = g(f(x)) \Rightarrow g(f(x)) \text{ is T-per.}$$

2.2.7 (a) $f(x) = |\sin x|$, $-\pi \le x < \pi$, 2π-per.

Fourier coeff:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin x|\,dx = \frac{2}{2\pi}\int_0^{\pi} \sin x\,dx$$
$$= -2\cos x\Big|_0^{\pi}$$
$$= \frac{4}{2\pi} = \boxed{\frac{2}{\pi}}$$

For $n > 0$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|\sin x| \cos nx}_{\text{even fun.}}\, dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$\overset{n\neq 1}{=} -\frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{2(-1)^{n+1}}{1-n^2} - \frac{2}{1-n^2}\right] = \begin{cases} \frac{4}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \quad (*)$$

Case $n=1$:

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin(1+1)x \, dx = -\frac{1}{\pi} \frac{\cos 2x}{2}\Big|_0^{\pi} = 0$$

So formula (*) also works for a_1.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|\sin nx| \sin nx}_{\text{odd}}\, dx = 0$$

$$\Rightarrow \boxed{f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{1-(2k)^2} \cos 2kx}$$

obtained by keeping only the non-zero terms in Fourier series. They correspond to: $n = 2k$

See attached plot and code.

2.2.9

(a) $f(x) = x^2$, if $-\pi \le x \le \pi$, 2π-per.

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\,\frac{2\pi^3}{3} = \frac{\pi^2}{3}}$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2 \sin nx}_{\text{odd fun}}\,dx = 0}$$

$$\boxed{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx \underset{\text{IBP}}{=} \underbrace{\frac{1}{\pi} x^2 \frac{\sin nx}{n}\Big|_{-\pi}^{\pi}}_{=0} - \frac{1}{\pi}\int_{-\pi}^{\pi} 2x\,\frac{\sin nx}{n}\,dx$$

$$\underset{\text{IBP}}{=} -\frac{2x}{\pi}\left(-\frac{\cos nx}{n^2}\right)\Big|_{-\pi}^{\pi} - \underbrace{\frac{2}{\pi}\int_{-\pi}^{\pi}\frac{\cos nx}{n^2}\,dx}_{=0 \text{ since } \int_0^{2\pi}\cos x\,dx = 0}$$

$$= \frac{1}{\pi}\,\frac{4\pi}{n^2}(-1)^n = \boxed{\frac{4}{n^2}(-1)^n}$$

$$\Rightarrow \boxed{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx}$$

(b) see attached code and plot.

2.2.10 (only part a)

(a) $f(x) = 1 - \sin x + 3\cos 2x$

using $\perp$ relations:

$$\boxed{a_0 = \frac{(f,1)}{(1,1)} = \frac{(1,1)}{(1,1)} = 1}$$

$n \geq 1$

$$\boxed{a_n = \frac{(f, \cos nx)}{(\cos nx, \cos nx)}} = \boxed{\begin{cases} \dfrac{3(\cos 2x, \cos 2x)}{(\cos 2x, \cos 2x)} & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}}$$

$$b_n = \frac{(f, \sin nx)}{(\sin nx, \sin nx)} = \begin{cases} -\frac{(\sin x, \sin x)}{(\sin x, \sin x)} = -1, & n=1 \\ 0 & \text{otherwise} \end{cases}$$ (4)

Thus $a_0 = 1$, $a_2 = 3$, $b_1 = -1$ and all other coeff are zero. Hence:

$$\boxed{f(x) = 1 - \sin x + 3\cos 2x.}$$

2.2.11

(a) $f(x) = \sin^2 x = \frac{1}{2}(1 - \cos 2x)$ (trig formulas)

using similar reasoning as above the only non-zero coeff in Fourier series are.

$$\boxed{a_0 = \frac{1}{2}, \quad a_2 = -\frac{1}{2}.}$$

$\Rightarrow$ result.

$$f(x) = \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\Rightarrow \quad \boxed{a_0 = \frac{1}{2}, \quad a_2 = \frac{1}{2}}$$

2.2.13

(a) $f(x) = x$ if $-\pi \le x \le \pi$

we have $f(x) = 2g(\pi - x)$ where g is def in example 1.

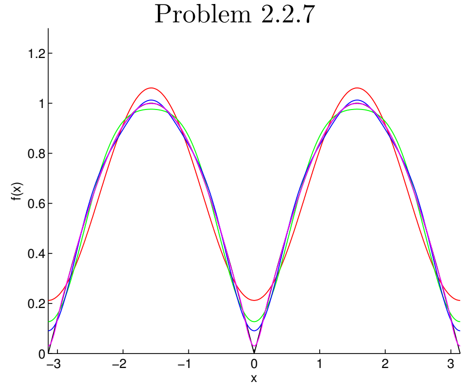
Thus:

$$f(x) = 2g(\pi - x) = 2\sum_{n=1}^{\infty} \frac{\sin n(\pi - x)}{n}$$

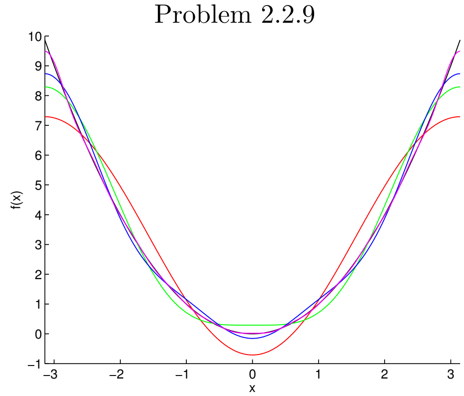
$$= -2 \sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n}$$

(b) plot & code included.

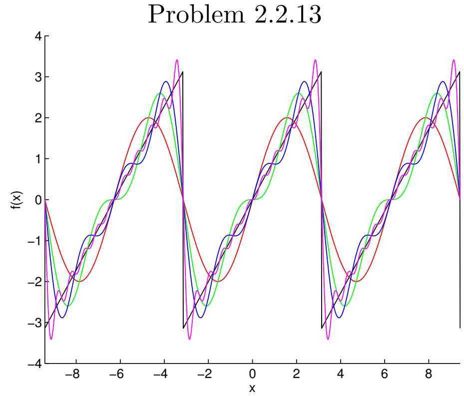
Problem 2.2.7



Problem 2.2.9



Problem 2.2.13



```
% math 3150 - problem 2.2.7
x = linspace(-3*pi,3*pi,1000);

figure(1); clf;
Ns = [1,2,3,10];
cols  = {'r','g','b','m'}; % use differnet colors

% plot true function
hold on;
plot(x,mod(x-pi,2*pi)-pi,'k'); % print in black
for iN=1:length(Ns),
 N = Ns(iN);
 % evaluate Fourier series up to N
 f = 0;
 for n=1:N,
  f = f + 2*(-1)^(n+1)*sin(n*x)/n;
 end;
 plot(x,f,cols{iN});
end;
hold off;
xlabel('x'); ylabel('f(x)');
axis([-3*pi,3*pi,-4,4]);
```



```
% math 3150 - problem 2.2.7
x = linspace(-pi,pi,1000);

figure(1); clf;
Ns = [1,2,3,10];
cols  = {'r','g','b','m'}; % use differnet colors

% plot true function
hold on;
plot(x,abs(sin(x)),'k'); % print in black
for iN=1:length(Ns),
 N = Ns(iN);
 % evaluate Fourier series up to N
 f = 2/pi;
 for k=1:N,
  f = f + (4/pi/(1-(2*k)^2)) * cos(2*k*x);
 end;
 plot(x,f,cols{iN});
end;
hold off;
xlabel('x'); ylabel('f(x)');
axis([-pi,pi,0,1.3]);
```

```
% math 3150 - problem 2.2.7
x = linspace(-pi,pi,1000);

figure(1); clf;
Ns = [1,2,3,10];
cols  = {'r','g','b','m'}; % use differnet colors

% plot true function
hold on;
plot(x,x.^2,'k'); % print in black
for iN=1:length(Ns),
 N = Ns(iN);
 % evaluate Fourier series up to N
 f = pi^2/3;
 for n=1:N,
  f = f + 4*(-1)^n*cos(n*x)/n^2;
 end;
 plot(x,f,cols{iN});
end;
hold off;
xlabel('x'); ylabel('f(x)');
axis([-pi,pi,-1,10]);
```