MATH 3150 - HW1 (S12).

SOLUTIONS

Prob 1.1.11

(a) We use method of characteristics to find sol to:

$$\frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 0$$

Characteristics must satisfy: $\frac{dy}{dx} = \frac{x^2}{1} \Rightarrow y = \frac{x^3}{3} + c.$

$$\Rightarrow \boxed{u(x,y) = f\left(y - \frac{x^3}{3}\right)}$$

Note: Since f is arbitrary it is OK to have $\frac{x^3}{3} - y$ as argument to f

(b)

$$\frac{\partial u}{\partial x} = -x^2 f'\left(y - \frac{x^3}{3}\right)$$

$$+x^2 \times \left[\frac{\partial u}{\partial y} = f'\left(y - \frac{x^3}{3}\right)\right]$$

$$\frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 0$$

Prob 1.2.3 We need to show that $u(x,t) = F(x+ct) + G(x-ct)$ solves wave eq.

$$\frac{\partial^2 u}{\partial t^2} = c^2 F''(x+ct) + (-c)(-c)\, G''(x-ct)$$

$$(-c^2) \times \left[\frac{\partial^2 u}{\partial x^2} = F''(x+ct) + G''(x-ct)\right]$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

This is called D'Alembert's solution to 1D WEQ.

Problem 1.2.5a)

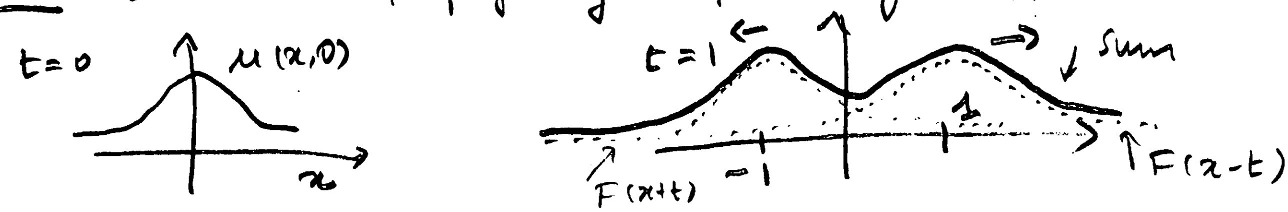
(a) Solve
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = \dfrac{1}{1+x^2} \\ \dfrac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

We have:
$$\boxed{u(x,0) = F(x) + G(x) = \frac{1}{x^2+1}} \quad (*)$$
$$\frac{\partial u}{\partial t}(x,t) = cF'(x+ct) - cG'(x-ct)$$
$$\Rightarrow \quad \frac{\partial u}{\partial t}(x,0) = cF'(x) - cG'(x) = 0$$
$$\Rightarrow \quad F'(x) = G'(x)$$
integration constant
$$\Rightarrow \quad \boxed{F(x) = G(x) + d} \quad (**)$$

Therefore: from $(*)$ and $(**)$ we get.

$$2F(x) + d = \frac{1}{x^2+1}$$
$$\Rightarrow \quad \boxed{F(x) = \frac{1}{2(x^2+1)} - \frac{d}{2}}$$
$$\boxed{G(x) = \frac{1}{2(x^2+1)} + \frac{d}{2}}$$
constants cancel out
$$\Rightarrow \quad u(x,t) = \frac{1}{2}\left[\frac{1}{(x+ct)^2+1} + \frac{1}{(x-ct)^2+1}\right]$$

Note: sol is a blob propagating in pos & neg dirs:



Problem 1.2.7

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = -2x e^{-x^2}$$

From previous problem:

$$\begin{cases} u(x,0) = F(x) + G(x) = 0 \\ \frac{\partial u}{\partial t}(x,0) = cF'(x) - cG'(x) = -2x e^{-x^2} \end{cases}$$

$$\Rightarrow \boxed{F(x) = -G(x)}$$

$$\Rightarrow \quad c2F'(x) = -2x e^{-x^2}$$

$$F'(x) = \frac{-2x}{2c} e^{-x^2}$$

$$F(x) = \frac{e^{-x^2}}{2c} + d$$

integration const will cancel out in final sol.

$$G(x) = -\frac{e^{-x^2}}{2c} - d$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2c}\left[e^{-(x+ct)^2} - e^{-(x-ct)^2}\right]}$$

Problem 1.2.16

We know that $u_n(x,t) = \sin\frac{n\pi x}{L}\cos\frac{n\pi c t}{L}$ solves 1DWEQ

with:

$$\begin{cases} u_n(x,0) = \sin\frac{n\pi x}{L} \\ \frac{\partial u_n}{\partial t}(x,0) = 0 \end{cases}$$

Thus if u solves 1DWEQ with:

$$\begin{cases} u(x,0) = \frac{1}{2}\sin\frac{\pi x}{L} + \frac{1}{4}\sin\frac{3\pi x}{L} \\ \frac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

Then:

$$\boxed{u(x,t) = \frac{1}{2}\underbrace{\sin\frac{\pi x}{L}\cos\frac{\pi c t}{L}}_{u_1(x,t)} + \frac{1}{4}\underbrace{\sin\frac{3\pi x}{L}\cos\frac{3\pi c t}{L}}_{u_3(x,t)}}$$

by principle of superposition (linearity).