Now $c_2 = 0$ since we do not want unbounded solutions for $r=0$.

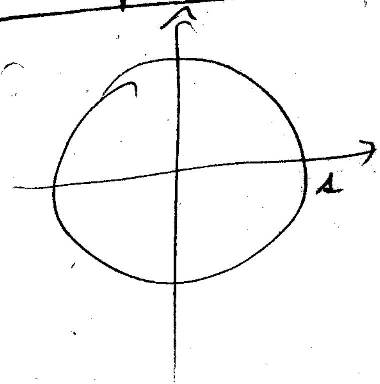
thus:

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[a_n \cos n\theta + b_n \sin n\theta\right]$$

$$\Rightarrow u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

$$\Rightarrow \left| \begin{array}{l} a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\, d\theta \\ a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \\ b_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta \end{array} \right.$$

Example 4.4.1



Steady state temp distrib on disk of radius 1 with $u(1,\theta) = f(\theta) = \begin{cases} 100 & 0<\theta<\pi \\ 0 & \pi<\theta<2\pi \end{cases}$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} 100\, d\theta = 50$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} 100 \cos n\theta \, d\theta = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} 100 \sin n\theta \, d\theta = -\frac{100}{n\pi} \cos n\theta \Big|_0^{\pi}$$

$$= \frac{100}{n\pi}\left(1 - \cos n\pi\right)$$

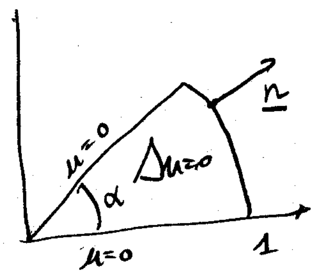
$$\Rightarrow u(r,\theta) = 50 + \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (1-\cos n\pi)\, r^n \sin n\theta$$

Of course by construction at $r=1$ we should get back Fourier series of f.

Other regions and Boundary conditions

With polar coordinates it is relatively easy to solve Laplace equation on annulus, wedges, or outside disk.

Example 4.4.4 Laplace equation on a wedge with Robin B.C.



$$\begin{cases} \Delta u = 0 & \text{on wedge} \\ u = 0 & \text{on sides of wedge} \\ \underline{n}\cdot\nabla u + u = -\theta & \text{on } r=1 \text{ boundary of wedge} \end{cases}$$

(Recall Robin B.C. are in general:
$\underline{n}\cdot\nabla u + a u = f$
$\underline{n}$ = unit outward pointing vector normal to boundary)

In our case $\underline{n}\cdot\nabla u\Big|_{r=1} = \dfrac{\partial u}{\partial n}\Big|_{r=1} = \dfrac{\partial u}{\partial r}(1,\theta)$

Thus B.C. on part of wedge where $r=1$ is:

$$u_r(1,\theta) + u(1,\theta) + \theta = 0$$

__Method__: separation of variables

$$u(r,\theta) = R(r)\,\Theta(\theta)$$

as before we get: (1) $\begin{cases} \Theta'' + \lambda^2\Theta = 0 \\ \Theta(0) = \Theta(\alpha) = 0 \end{cases}$

and (2) $r^2 R'' + rR' - \lambda^2 R = 0$

Solving (1) we get $\Theta(\theta) = a\cos\lambda\theta + b\sin\lambda\theta$, but $\Theta(0)=0 \Rightarrow a=0$
and $\Theta(\alpha)=0 \Rightarrow \lambda = \lambda_n = \dfrac{n\pi}{\alpha}$

$$\Rightarrow \boxed{\Theta_n = \sin\frac{n\pi}{\alpha}\theta} \qquad n=1,2,\dots$$

Now we get for (2):

$$r^2R'' + rR' - \left(\frac{n\pi}{\alpha}\right)^2 R = 0$$

Solution is of the form $R(r) = a\, r^{\frac{n\pi}{\alpha}} + b r^{-\frac{n\pi}{\alpha}}$

But since $R(0)$ must be bounded we have $b = 0$

$$\Rightarrow \quad R_n(r) = r^{\frac{n\pi}{\alpha}}$$

and $u_n(r,\theta) = b_n\, r^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\theta\right)$

solves DE by construction.

$\Rightarrow$ Solution is of the form:

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} b_n\, r^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi}{\alpha}\theta\right)}, \quad r \in [0,1], \ \theta \in [0,\alpha].$$

To find b_n we look at B.C.:

$$u_r(r,\theta) = \sum_{n=1}^{\infty} b_n \frac{n\pi}{\alpha} r^{\left(\frac{n\pi}{\alpha}-1\right)} \sin\left(\frac{n\pi}{\alpha}\theta\right)$$

$$u_r(1,\theta) = \sum_{n=1}^{\infty} b_n \frac{n\pi}{\alpha} \sin \frac{n\pi}{\alpha}\theta$$

$$\underset{\substack{\uparrow \\ \text{Robin B.C.}}}{=} -u(1,\theta) - \theta$$

$$= -\theta - \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{\alpha}\theta.$$

$$\Rightarrow \quad -\theta = \sum_{n=1}^{\infty} \underbrace{b_n\left(1 + \frac{n\pi}{\alpha}\right)} \sin \frac{n\pi}{\alpha}\theta$$

Coeff in half range sine series of $-\theta$:

$$b_n\left(1+\frac{n\pi}{\alpha}\right) = -\frac{2}{\alpha}\int_0^{\alpha} \theta \sin \frac{n\pi}{\alpha}\theta \cdot d\theta$$

$$= -\frac{2}{\alpha}\left[-\theta \frac{\alpha}{n\pi} \cos \frac{n\pi}{\alpha}\theta \Big|_0^{\alpha} + \frac{\alpha}{n\pi}\int_0^{\alpha} \cos \frac{n\pi}{\alpha}\theta \, d\theta\right]$$

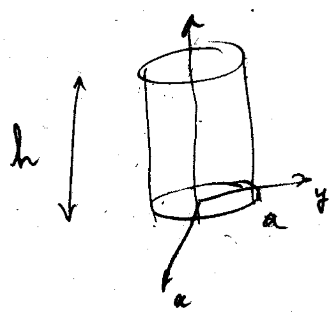
$$= \frac{2\alpha(-1)^n}{n\pi}$$

To conclude:

$$\boxed{u(r,\theta) = \frac{2\alpha^2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n(\alpha + n\pi)}\, r^{\frac{n\pi}{\alpha}} \sin \frac{n\pi}{\alpha}\theta .}$$

§ 4.5 Laplace Equation in a Cylinder

Recall cylindrical coordinates (~ polar in xy plane + height z)



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\Delta u = u_{xx} + u_{yy} + u_{zz}$$
$$= \underbrace{u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}}_{\text{Laplacian in polar coord.}} + u_{zz}$$

The domain we work on is a cylinder defined by:

$0 \le \theta \le 2\pi$, $0 \le r \le a$, $0 \le z \le h$.

Consider first the case where:

- bottom of cylinder is kept at temp 0
- lateral sides of cylinder ______
- top part of cylinder has a radially symmetric temperature

Then the PDE we would like to solve is:

$$\begin{cases} \Delta u = 0 & \text{inside cylinder} \\ u(r,0) = 0 & 0 \le r \le a \\ u(a,z) = 0 & 0 \le z \le h \\ u(r,h) = f(r) & 0 \le r \le a \end{cases}$$

Separation of variables $u(r,z) = R(r)\,Z(z)$

$$(1)\begin{cases} r^2 R'' + rR' - k r^2 R = 0 \\ R(a) = 0 \end{cases} \qquad (2)\begin{cases} Z'' + kZ = 0 \\ Z(0) = 0 \end{cases}$$

Solving (1) (slightly more complicated than circular case)

$\underline{k=0}$

$$\begin{cases} rR'' + R' = 0 \Rightarrow R(r) = \alpha \ln r \\ R(a) = 0 \Rightarrow R(r) = 0. \end{cases}$$

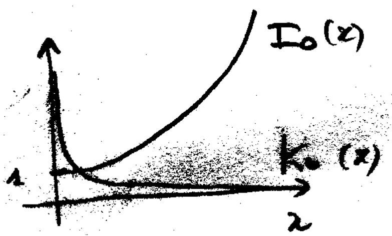
$\underbrace{\text{blows up at } r=0}$

$\underline{k=-\lambda^2<0}$ sol to (1) is $R(r) = \alpha J_0(\lambda r) + \beta Y_0(\lambda r)$

$R(0)=0 \Rightarrow \beta = 0.$

$\Rightarrow \boxed{R_n(r) = J_0\left(\frac{\alpha_n}{a} r\right)}$, $\alpha_n =$ n-th zero of $J_0(\cdot)$

$\underline{k=\lambda^2>0}$ sol to (1) is $R(r) = \alpha I_0(\lambda r) + \beta K_0(\lambda r)$



K_0 is unbounded at $r=0$ thus $R(a)=0 \Rightarrow \beta = 0$

I_0 grows and $I_0(0)=1 \to \alpha = 0.$

$I_0 =$ modified Bessel function of the first kind

$K_0 =$ " " " " second kind

$i^p I_p(x) = J_p(ix)$ $\quad (\sim \sin ix = i \sinh x)$

$K_p(x) = \frac{\pi}{2\sin p\pi}\left[I_{-p}(x) - I_p(x)\right]$ ($\sim$ same way Y was obtained from J)

$p=0$ case obtained as $p \to 0$ above.

Bessel functions J_p, Y_p solve:

$$r^2 y'' + r y' + (r^2 - p^2) y = 0$$

do change of vars: $r = ix$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} = -i\frac{\partial}{\partial x}$$

$$(ix)^2(-i)^2\frac{\partial^2 y}{\partial x^2} + (ix)(-i)\frac{\partial y}{\partial x} + ((ix)^2 - p^2)\,y = 0$$

Modified Bessel fun I_p, K_p solve

$$r^2 y'' + r y' - (r^2 + p^2) y$$

Thus only case $k=-\lambda^2$ is interesting

$$Z_n(z) = \alpha \cosh \lambda_n z + \beta \sinh \lambda_n z$$

$$Z_n(0) = \alpha = 0 \Rightarrow \boxed{Z_n(z) = \sinh \lambda_n z}$$

Thus solution is of the form:

$$\boxed{u(r,z) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \sinh \lambda_n z}$$

B.C.:

$$u(r,h) = f(r) = \sum_{n=1}^{\infty} \underbrace{A_n \sinh(\lambda_n h)}_{\text{Bessel series coeff of } f(r)} J_0(\lambda_n r)$$

$$\rightarrow \boxed{A_n = \frac{1}{\sinh(\lambda_n h)} \frac{2}{a^2 J_1(\alpha_n)} \int_0^r J_0(\lambda_n r) f(r) r\,dr}$$

Dirichlet problem with non zero lateral temp

Very similar derivation, however case $k=\lambda^2$ is the interesting one

$$\begin{cases} rR'' + R' - kr^2 R = 0 \\ R(a) = 1 \end{cases} \qquad \begin{cases} Z'' + kZ = 0 \\ Z(0) = Z(h) = 0 \end{cases}$$

$$\rightarrow R_n(r) = I_0\left(\frac{n\pi}{h} r\right) \qquad \Rightarrow Z_n(z) = \sin \frac{n\pi}{h} z$$

$$\Rightarrow u(r,z) = \sum_{n=1}^{\infty} B_n I_0\left(\frac{n\pi}{h} r\right) \sin \frac{n\pi}{h} z$$

B.C.

$$f(z) = u(a,z) = \sum_{n=1}^{\infty} \underbrace{B_n I_0\left(\frac{n\pi}{h} a\right)}_{\text{coeff in sine series}} \sin \frac{n\pi}{h} z.$$

$$\Rightarrow B_n = \frac{1}{I_0\left(\frac{n\pi a}{h}\right)} \frac{2}{h} \int_0^h f(z) \sin \frac{n\pi}{h} z\,dz$$