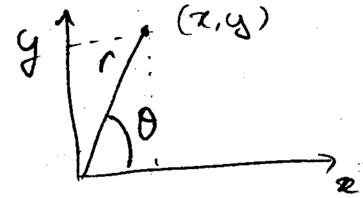
§4 PDE in polar and cylindrical coordinates

- **Polar coordinates**

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \leftrightarrow \begin{cases} r^2 = x^2+y^2 \\ \tan\theta = \frac{x}{y} \end{cases}$$



$\theta = \tan^{-1}\frac{y}{x} + k\pi$,
where k depends on x and y.

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

(all terms have same units as r^{-2}).

is the representation of Laplacian in polar coordinate.
Deriving is long and tedious but it depends only on using chain rule and knowing derivatives such as:

$r = \sqrt{x^2+y^2} \rightarrow$ $\boxed{\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta}$, $\boxed{\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta}$

$\theta = \tan^{-1}\frac{y}{x} \rightarrow$ $\boxed{\frac{\partial\theta}{\partial x}} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2} = \boxed{-\frac{\sin\theta}{r}}$

$\boxed{\frac{\partial\theta}{\partial y}} = \frac{1}{1+(y/x)^2}\frac{1}{x} = \frac{x}{r^2} = \boxed{\frac{\cos\theta}{r}}$

$$\boxed{\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}(\cos\theta) = -\frac{\partial\theta}{\partial x}\sin\theta = +\frac{\sin^2\theta}{r}}$$

$$\boxed{\frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y}(\sin\theta) = \frac{\partial\theta}{\partial y}\cos\theta = \frac{\cos^2\theta}{r}}$$

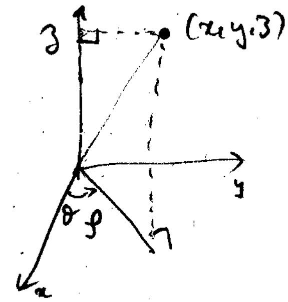
$$\boxed{\frac{\partial^2\theta}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\sin\theta}{r}\right) = -\frac{\partial}{\partial x}(\sin\theta)\frac{1}{r} - \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\sin\theta = 2\frac{\cos\theta\sin\theta}{r^2}}$$

$$\boxed{\frac{\partial^2\theta}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\cos\theta}{r}\right) = \frac{\partial}{\partial y}(\cos\theta)\frac{1}{r} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\cos\theta = -2\frac{\cos\theta\sin\theta}{r^2}}$$

(these are included to practice with changes of coordinates, see p195 for more details)

- Cylindrical coordinates

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases}$$



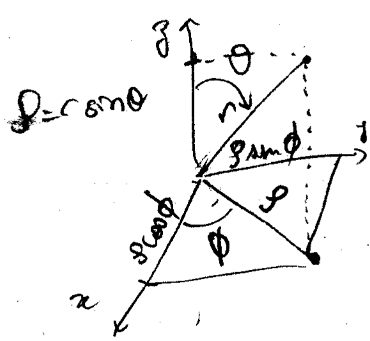
In cartesian coordinates: $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

using polar form of Laplacian:

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}}$$

- Spherical coordinates

$$\begin{cases} x = r\cos\phi\sin\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\theta \end{cases}$$



we have:
$r^2 = x^2 + y^2 + z^2$

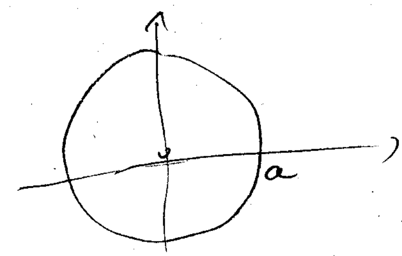
$$\boxed{\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right)}$$

Derivation involves using Laplac polar form on xy-plane and $z\rho$ plane, where ρ is projection of (x,y,z) onto (x,y) plane.

see p137. We are not going to cover §5 so we won't use this too much in this class.

§ 4.2 Vibrations of a circular membrane (radially symm)

We study vibrations of a circular membrane (drum) clamped on edges i.e. $u(a,\theta,t) = 0$, where $u(r,\theta,t) =$ displacement of membrane from equilibrium.



These are governed by the 2D wave equation:

$$u_{tt} = c^2 \Delta u$$

which in polar coordinates (better adapted to this problem than cartesian)

$$\boxed{u_{tt} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)}$$

We consider first the case where:

initial shape $f(r,\theta) = f(r)$ (radially symm)

initial velocity $g(r,\theta) = g(r)$

Because of the symmetries of the problem, $u(r,\theta) = u(r)$

$$\Rightarrow \frac{\partial u}{\partial \theta} = 0$$

Thus we are left with the simplified PDE:

$$\begin{cases} u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r \right) \\ u(a,t) = 0 \\ u(r,0) = f(r) \\ u_t(r,0) = g(r) \end{cases}$$

Use separation of variables: $u(r,t) = R(r)T(t)$

$$RT'' = c^2 \left(R''T + \frac{1}{r} R'T \right)$$

$$\Rightarrow \frac{T''}{c^2 T} = \frac{1}{R}\left(R''T + \frac{1}{r} R' \right) = -\lambda^2 = \text{const}$$

otherwise no time periodic sol!

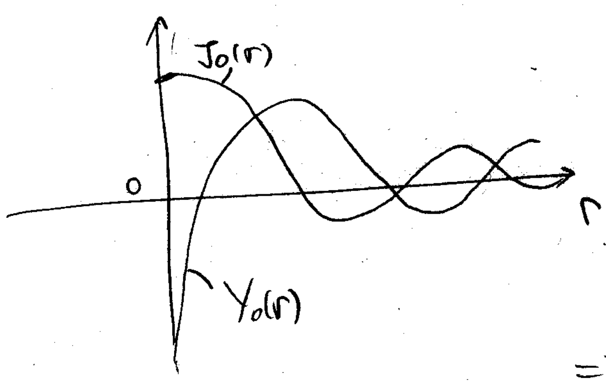
We get 2 equations to solve:

$$(1)\ \begin{cases} rR'' + R' + \lambda^2 r R = 0 \\ R(a) = 0 \end{cases} \qquad (2)\ \begin{cases} T'' + c^2\lambda^2 T \\ \dots \end{cases}$$

(1) Bessel equation of order 0 (parameter form)
2nd order linear eq $\Rightarrow$ we need 2 lin indep sol to describe all possible solutions.

- $J_0(\lambda r)$ Bessel function of order 0 of first kind
- $Y_0(\lambda r)$ ——— 2nd ———

Sol to (1) is
$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$$



$r \to 0 \Rightarrow Y_0(r) \to -\infty$
Since $|R(0)| < \infty$
we must have $c_2 = 0$

$\Rightarrow R(r) = c_1 J_0(\lambda r)$, take $c_1 = 1 \neq 0$
$R(a) = c_1 J_0(\lambda a) = 0$

λa must be a root of Bessel fun J_0.

Let $\alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$ be zeros of J_0

then $\boxed{\lambda = \lambda_n = \frac{\alpha_n}{a}}$, $n = 1, 2 \dots$

and $\boxed{R_n(r) = J_0\left(\frac{\alpha_n}{a} r\right)}$, $n = 1 \dots$

(analogous to the ones in Sine Series expansion:
we solve $\sin(\lambda L) = 0 \Leftrightarrow \lambda L = n\pi$
$\lambda_n = \frac{n\pi}{L}$,

and analogy does not stop here!

Then we go back to (2) and write its solution:

$$T_n(t) = A_n \cos\left(\frac{c\,\alpha_n}{a} r\right) + B_n \sin\left(\frac{c\,\alpha_n}{a} r\right)$$

→ By construction:
$$\boxed{u_n(r,t) = R_n(t)T_n(t) = \left(A_n \cos\left(\frac{c\,\alpha_n}{a} r\right) + B_n \sin\left(\frac{c\,\alpha_n}{a} r\right)\right) J_0\left(\frac{\alpha_n}{a} r\right)}$$

$n = 1, 2, \ldots$

solves DE and so does:

$$u(r,t) = \sum_{n=1}^{\infty} \left(A_n \cos(c\lambda_n t) + B_n \sin(c\lambda_n t)\right) J_0(\lambda_n r)$$

What about B.C.?

$$f(r) = u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \quad (*) \quad = \text{Bessel Series Expansion}$$

Looks a lot like Fourier series expansion, and it relies on $J_0(\lambda_n r)$ being orthogonal in the inner prod:

$$(u,v) = \int_0^a u(r)v(r)\, r\,dr \qquad \left(\text{compare to } (u,v) = \int_{-p}^{p} u(x)v(x)dx\right.$$

and:
$$\left(J_0(\lambda_n r), J_0(\lambda_m r)\right) = \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{a^2 J_1^2(\alpha_n)}{2} & \text{if } n = m \end{cases}$$

thus: dotting by $J_0(\lambda_n r)$ on both sides of $(*)$

$$A_n = \frac{(f(r), J_0(\lambda_n r))}{(J_0(\lambda_n r), J_0(\lambda_n r))} = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r) J_0(\lambda_n r)\, r\,dr$$

here $J_1(r)$ = Bessel fn of order 1 (available in matlab ...

· Initial velocity gives:

$$u_t(r,0) = g(r) = \sum_{n=1}^{\infty} c\lambda_n B_n J_0(\lambda_n r)$$

$$\Rightarrow \quad c\lambda_n B_n = \frac{(g(r), J_0(\lambda_n r))}{(J_0(\lambda_n r), J_0(\lambda_n r))}$$

$$B_n = \frac{1}{c\lambda_n} \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a g(r) J_0(\lambda_n r)\, r\, dr.$$

Bessel functions

Bessel equation of order $p \geq 0$:

$$r^2 y'' + r y' + (r^2 - p^2) y = 0, \qquad x > 0. \qquad \text{(BE)}$$

The general sol to (BE) of order p is:

$$y(r) = c_1 J_p(r) + c_2 Y_p(r)$$

If p is not an integer then a general sol is:

$$y(r) = c_1 J_p(r) + c_2 J_{-p}(r)$$

$J_p(r)$: Bessel function of order p of the first kind

$Y_p(r)$ = " " " p " " second kind

It is possible to derive series representations for Bessel functions:

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p}$$

$$Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}, \quad p \text{ not an integer}$$

$n \in \mathbb{N}$ $Y_n(x)$ is constructed by a lim process: $Y_n(x) = \lim_{p \to n} Y_p(x)$

parenthesis ↑

Here $\Gamma(x)$ is the Gamma function.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$ (you don't need to know this)

The important properties of this function is that it behaves like the factorial and actually coincides with it at integer values:

$$\Gamma(x+1) = x\,\Gamma(x)$$ (analogous to $(n+1)! = (n+1)n!$

$$\Gamma(n+1) = n!$$, n integer

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$ (works even when factorial is not defined)

Back to Bessel functions: when $p = n \in \mathbb{N}$:

$$\boxed{J_{-n}(r) = (-1)^n J_n(r)}$$

Other properties of Bessel functions:

- If $p > 0$ $J_p(0) = 0$
- $\frac{d}{dr}\left[r^p J_p(r)\right] = r^p J_{p-1}(r)$ (~ recursion)
- $\frac{d}{dr}\left[r^{-p} J_p(r)\right] = -r^{-p} J_{p+1}(r)$

<u>Orthogonality of Bessel functions</u> (what makes them nice functions for doing expansions)

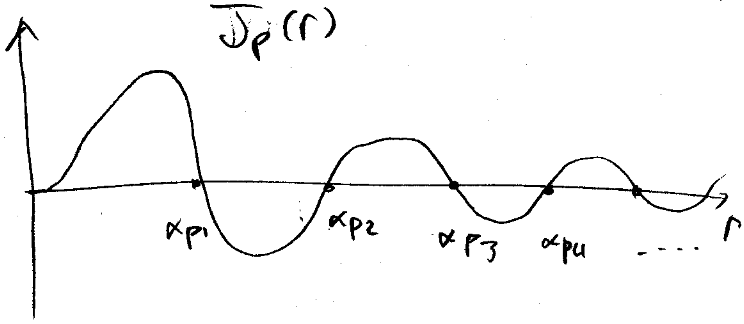
We can form a family of $\perp$ functions from J_p as follows:

Consider the inner product: ↙ weighted inner prod

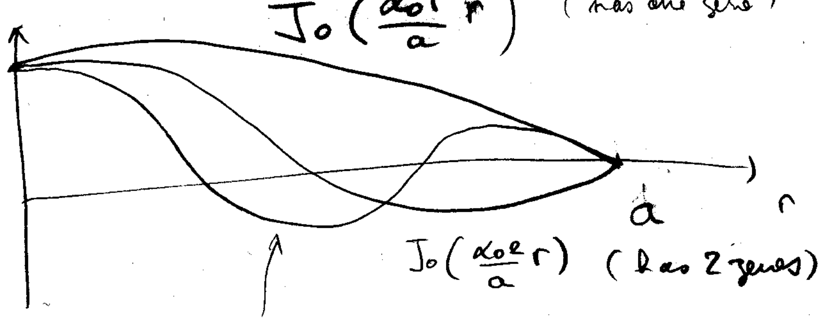
$$(u, v) = \int_0^a u(r)\, v(r)\, r\, dr$$

then:

$$\left(J_p\left(\frac{\alpha_{pj}}{a} x\right), J_p\left(\frac{\alpha_{pk}}{a} x\right)\right) = \begin{cases} 0 & \text{if } k \neq j \\ \frac{a^2}{2} J_{p+1}^2(\alpha_{pj}) & \text{otherwise} \end{cases}$$



then the orthogonal Bessel functions are dilated s.t. they are zero at a: For $J_0(r)$ we get.



<u>Bessel series expansion</u>

If f is piece smooth on $[0, a]$

$$f(r) = \sum_{j=1}^{\infty} A_j J_p\left(\frac{\alpha_{pj}}{a} r\right)$$

where

$$A_j = \frac{\left(f(r), J_p\left(\frac{\alpha_{pj}}{a} r\right)\right)}{\left(J_p\left(\frac{\alpha_{pj}}{a} r\right), J_p\left(\frac{\alpha_{pj}}{a} r\right)\right)}$$

Thus
$$\boxed{A_j = \frac{2}{a^2 J_{p+1}^2(\alpha_{pj})} \int_0^a f(r)\, J_p\left(\frac{\alpha_{pj}}{a}\right) \underset{p}{\mathbf{r}}\, dr}$$

⚠ do not forget that this inner product is WEIGHTED.

Thus $$\boxed{A_j = \frac{2}{a^2 J_{p+1}^2(\alpha_{pj})} \int_0^a f(r)\, J_p\left(\frac{\alpha_{pj}}{a}\right) r\, dr}$$

⚠ do not forget that this inner product is WEIGHTED.

Recall that after using polar coordinates and using symmetries of problem, we get the simplified PDE that describes vibrations of a membrane with radially symmetric initial conditions:

$$\begin{cases} u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right) \\ u(a,t) = 0 \\ u(r,0) = f(r) \\ u_t(r,0) = g(r) \end{cases}$$

Using separation of variables we reach the solution:

$$u(r,t) = \sum_{n=1}^{\infty} \left(A_n \cos c\lambda_n t + B_n \sin c\lambda_n t\right) J_0(\lambda_n r)$$

where $J_0(\cdot)$ = Bessel function of order 0 of the first kind

$\lambda_n = \frac{\alpha_n}{a}$, α_n = n-th zero of J_0 (i.e. $J_0(\alpha_n) = 0$)

The A_n and B_n come from initial pos and init vel resp.

$$A_n = \frac{(f(r), J_0(\lambda_n r))}{(J_0(\lambda_n r), J_0(\lambda_n r))} = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r)\, J_0(\lambda_n r)\, r\, dr$$

$$c\lambda_n B_n = \frac{(g(r), J_0(\lambda_n r))}{(J_0(\lambda_n r), J_0(\lambda_n r))} = \text{''} \int_0^a g(r)\, J_0(\lambda_n r)\, r\, dr$$

Here is a particular example (example 4.2.2)

$$f(r) = 1 - r^2$$
$$g(r) = 0 \implies B_n = 0$$

$$A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 (1-r^2) J_0(\alpha_n r)\, r\, dr$$

$s = \alpha_n r$
$ds = \alpha_n dr$

$$= \frac{2}{J_1^2(\alpha_n)} \int_0^{\alpha_n} \left(1 - \left(\frac{s}{\alpha_n}\right)^2\right) J_0(s) \frac{s}{\alpha_n} \frac{ds}{\alpha_n}$$

$$= \frac{2}{\alpha_n^4 J_1^2(\alpha_n)} \int_0^{\alpha_n} (\alpha_n^2 - s^2) J_0(s)\, s\, ds$$

IBP: $(\alpha_n^2 - s^2)' = -2s$

$\int J_0(s)\, s\, ds \doteq J_1(s)\, s$ (see p 47 of notes)

$$= \frac{2}{\alpha_n^4 J_1^2(\alpha_n)} \left[(\alpha_n^2 - s^2) J_1(s) s \Big|_0^{\alpha_n} + 2 \int_0^{\alpha_n} J_1(s) s^2 ds \right]$$

$$= \frac{4}{\alpha_n^4 J_1^2(\alpha_n)} \int_0^{\alpha_n} J_1(s) s^2 ds$$

again from p 47 in notes

$$= \frac{4}{\alpha_n^4 J_1^2(\alpha_n)} J_2(s) s^2 \Big|_0^{\alpha_n} = \boxed{\frac{4}{\alpha_n^2 J_1^2(\alpha_n)} J_2(\alpha_n)}$$

Show Matlab version

§4.3 Vibrations of circular membrane (General case)

This time we do not assume I.C. are radially symmetric. We want to solve:

$$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right) = c^2 \Delta u \quad \text{(in polar coord)}$$

$$\begin{cases} u(r,\theta,0) = f(r,\theta) \\ u_t(r,\theta,0) = g(r,\theta) \end{cases} \text{ Initial conditions}$$

$$u(a,\theta,t) = 0 \quad \text{Boundary conditions}$$

Solution using separation of variable: $u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$

$$\frac{T''}{c^2T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\lambda^2 = \text{constant}$$

(sign to get periodic sol. in T, which is what physics tells us)

(1) $\dfrac{T''}{c^2T} = -\lambda^2$ and $\dfrac{R''}{R} + \dfrac{R'}{rR} + \dfrac{\Theta''}{r^2\Theta} = -\lambda^2$

Separation of variables again

(2) $\lambda^2 r^2 + \dfrac{r^2 R''}{R} + r\dfrac{R'}{R} = \mu^2$

(3) $-\dfrac{\Theta''}{\Theta} = \mu^2$ (sign because we know $u(r,\theta,t)$ should be 2π-per in θ: $u(r,\theta+2\pi,t) = u(r,\theta,t)$)

(3): Using periodicity of Θ and Θ':

$$\begin{cases} \Theta'' + \mu^2\Theta = 0 \\ \Theta(0) = \Theta(2\pi) = 0 \\ \Theta'(0) = \Theta(2\pi) = 0 \end{cases} \Rightarrow \Theta_m = A_m \cos m\theta + B_m \sin m\theta$$

for $m = 0, 1, 2, \dots$

(2): $\begin{cases} r^2R'' + rR' + (\lambda^2 r^2 - m^2)R = 0 \\ R(a) = 0 \end{cases}$

We saw that $J_m(x)$, $Y_m(x)$ solve:

$x^2y'' + xy' + (x^2 - m^2)y = 0$, which is almost like (2).

Letting $x = \lambda r$ and $R(r) = y(\lambda r)$,

$$\Rightarrow R'(r) = \lambda y'(\lambda r)$$
$$R''(r) = \lambda^2 y''(\lambda r)$$

$\Rightarrow (\lambda r)^2 \dfrac{R''}{\lambda^2} + \lambda r \dfrac{R'}{\lambda} + (\lambda^2 r^2 - m^2)R = 0$ which is (2)

Therefore:

$$R(r) = c_1 J_m(\lambda r) + \cancel{c_2 Y_m(\lambda r)}$$

$c_2 = 0$ because
$Y_m(r) \to -\infty$ as $r \to 0$.

$$\Rightarrow \quad R(r) = J_m(\lambda r)$$

using boundary conditions:

$$R(a) = J_m(\lambda a) = 0 \quad \Rightarrow \quad \lambda_{mn} = \frac{\alpha_{mn}}{a} \qquad \left(\begin{matrix} m = 0, 1, 2 \ldots \\ n = 1, 2, 3 \ldots \end{matrix}\right)$$

where α_{mn} is the nth zero of $J_m(r)$.

$$\Rightarrow \boxed{R_{mn}(r) = J_m(\lambda_{mn} r)}$$

Now (1) gives solutions of the form $a \cos c\lambda_{mn} t + b \sin c\lambda_{mn} t$

thus the solution has the form:

$$u(r,\theta,t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn} \cos m\theta + b_{mn} \sin m\theta\right) \cos c\lambda_{mn} t$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a^*_{mn} \cos m\theta + b^*_{mn} \sin m\theta\right) \sin c\lambda_{mn} t$$

Here a_{mn}, b_{mn} correspond to $f(r,\theta)$
and a^*_{mn}, b^*_{mn} — $g(r,\theta)$

How to get these ? a_{mn} and b_{mn}: (*)

$$u(r,\theta,0) = \boxed{f(r,\theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn} \cos m\theta + b_{mn} \sin m\theta\right)}$$

~ generalized Fourier series for f

To get coefficients, freeze r and look at (*) as if it where a Fourier series:

$$f(r,\theta) = \underbrace{\sum_{n=1}^{\infty} a_{0n} J_0(\lambda_{0n} r)}_{= a_0(r)}$$

$$+ \sum_{m=1}^{\infty} \left[\underbrace{\left\{ \sum_{n=1}^{\infty} a_{mn} J_m(\lambda_{mn} r) \right\}}_{= a_m(r)} \cos m\theta + \underbrace{\left\{ \sum_{n=1}^{\infty} b_{mn} J_m(\lambda_{mn} r) \right\}}_{= b_m(r)} \sin m\theta \right]$$

$$= a_0(r) + \sum_{m=1}^{\infty} a_m(r) \cos m\theta + b_m(r) \sin m\theta$$

$$\Rightarrow \quad a_0(r) = \frac{(f(r,\theta), 1)}{(1, 1)} = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta)\, d\theta$$

$$a_m(r) = \frac{(f(r,\theta), \cos m\theta)}{(\cos m\theta, \cos m\theta)} = \frac{1}{\pi} \int_0^{2\pi} f(r,\theta) \cos m\theta\, d\theta$$

$$b_m(r) = \frac{(f(r,\theta), \sin m\theta)}{(\sin m\theta, \sin m\theta)} = \frac{1}{\pi} \int_0^{2\pi} f(r,\theta) \sin m\theta\, d\theta$$

But each $a_m(r)$ or $b_m(r)$ is a Bessel series thus:

$$a_{mn} = \frac{(a_0(r), J_m(\lambda_{mn} r))}{(J_m(\lambda_{mn} r), J_m(\lambda_{mn} r))}$$

$$= \frac{2}{a^2 J_{m+1}^2(\alpha_{mn})} \int_0^a a_m(r) J_m(\lambda_{mn} r)\, r\, dr$$

$$= \begin{cases} \dfrac{1}{2\pi} \dfrac{2}{a^2 J_1^2(\alpha_{0n})} \displaystyle\int_0^a r\,dr \int_0^{2\pi} d\theta\, f(r,\theta) J_0(\lambda_{0n} r) & (m=0) \\[2ex] \dfrac{1}{\pi} \dfrac{2}{a^2 J_{m+1}^2(\alpha_{mn})} \displaystyle\int_0^a r\,dr \int_0^{2\pi} d\theta\, f(r,\theta) J_m(\lambda_{mn} r) \cos m\theta & (m>0) \end{cases}$$

And similarly:

$$b_{mn} = \frac{1}{\pi}\frac{2}{a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a r\,dr\int_0^{2\pi} d\theta\, f(r,\theta)\, J_m(\lambda_{m,n} r)\sin m\theta$$

$m = 1,2,\ldots$
$n = 1,2,\ldots$

It's a good exercise to check that: (w/ similar reasoning)

$$a_{0n}^* = \frac{1}{c\alpha_{0n}}\frac{1}{2\pi}\frac{2}{a^2 J_1^2(\alpha_{0n})}\int_0^a r\,dr\int_0^{2\pi} d\theta\, f(r,\theta)\, J_0(\lambda_{0n} r)$$

$$a_{mn}^* = \frac{1}{c\alpha_{mn}}\frac{1}{\pi}\frac{2}{a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a r\,dr\int_0^{2\pi} d\theta\, f(r,\theta)\, J_m(\lambda_{mn} r)\cos m\theta$$

$$b_{mn}^* = \frac{1}{c\alpha_{mn}}\frac{1}{\pi}\frac{2}{a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a r\,dr\int_0^{2\pi} d\theta\, f(r,\theta)\, J_m(\lambda_{mn} r)\sin m\theta$$

See book's examples

§4.4 Laplace's Equation in circular regions

$$\begin{cases} \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(a,\theta) = f(\theta) \qquad 0<\theta<2\pi \end{cases}$$

Laplace eq. w/ Dirichlet B.C.

Separation of variables:

$u(r,\theta) = R(r)\Theta(\theta)$ pluging into DE we get:

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

sign chosen s.t. Θ is periodic

$$\Theta'' + m\Theta = 0 \Rightarrow \Theta_n(\theta) = a_n\cos n\theta + b_n\sin n\theta \quad n=0,1,2\ldots$$

$$r^2R'' + rR' - n^2R = 0$$

App. A3 or Variation of param. method?

$$\begin{cases} R(r) = c_1\left(\frac{r}{a}\right)^n + c_2\left(\frac{r}{a}\right)^{-n} \\ R(r) = c_1 + c_2\ln\left(\frac{r}{a}\right). \end{cases}$$

Now $c_2 = 0$ since we do not want unbounded solutions for $r=0$.

thus:

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[a_n \cos n\theta + b_n \sin n\theta\right].$$

$$\Rightarrow u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

$$\Rightarrow \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$