Example 3.6.3

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 0 \\ u_x(L,t) = -u(L,t) \\ u(x,0) = f(x) = x(1-x) \end{cases}$$

see matlab code, as we cannot compute such roots explicitly.

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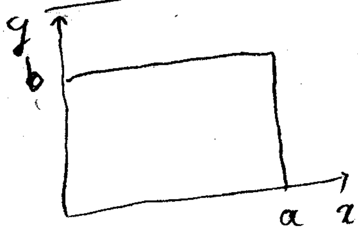
§ 3.7 The Wave and Heat equation in 2D

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{recall Laplacian}$$

The 2D Wave Equation:



$$\begin{cases} u_{tt} = c^2 \Delta u & 0 < x < a,\ 0 < y < b,\ t > 0 \\ u(0,y,t) = u(a,y,t) = 0 & 0 < y < b \\ u(x,0,t) = u(x,b,t) = 0 & 0 < x < a \\ u(x,y,0) = f(x,y) & 0 < x < a,\ 0 < y < b \\ u_t(x,y,0) = g(x,y) & 0 < x < a,\ 0 < y < b \end{cases}$$

B.C. / I.C.

models vibrations of a membrane on a rectangle
(we shall do other shapes in § 4)
$u(x,y,t)$ = displacement from equilibrium of point (x,y) at time t.

Solution using separation of variables

$$u(x,y,t) = X(x)Y(y)T(t) \quad \text{ansatz}$$

plugin into DE:

$$XYT'' = c^2(X''YT + XY''T)$$

$$\Rightarrow \underbrace{\frac{T''}{c^2 T}}_{F(t)} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{G(x,y)} = -k^2$$

(we use physical intuition: in time sol must be periodic so this rules out other cases. We could also be more rigorous and check other cases give trivial sol.)

thus we get:

$$T'' + k^2 c^2 T = 0$$

and $\underbrace{\frac{X''}{X}}_{\text{dep on } x} = \underbrace{-\frac{Y''}{Y} - k^2}_{\text{depends on } y} = \text{const}$

$$\begin{cases} \frac{X''}{X} = -\mu^2 \\ -\frac{Y''}{Y} - k^2 = -\mu^2 \end{cases}$$

$$\frac{Y''}{Y} = -(k^2 - \mu^2) = -\nu^2$$

Thus we need to solve:

$$\begin{cases} X'' + \mu^2 X = 0 \\ X(0) = X(a) = 0 \end{cases} \Rightarrow X(x) = c_1 \cos \mu x + c_2 \sin \mu x + \text{B.C.} \Rightarrow \mu = \mu_m = \frac{m\pi}{a} \quad m \geq 1$$

$$\begin{cases} Y'' + \nu^2 Y = 0 \\ Y(0) = Y(b) = 0 \end{cases} \Rightarrow Y(y) = d_1 \cos \nu y + d_2 \sin \nu y + \text{B.C.} \Rightarrow \nu = \nu_n = \frac{n\pi}{b} \quad n \geq 1$$

$$\Rightarrow \boxed{X_m(x) = \sin \frac{m\pi}{a} x \,, \quad Y_n(y) = \sin \frac{n\pi}{b} y}$$

For $T(t)$: we have $k_{m,n}^2 = \mu_m^2 + \nu_n^2$

$$\Rightarrow \boxed{T_{m,n}(t) = B_{mn} \cos(c k_{m,n} t) + B_{mn}^* \sin(c k_{m,n} t)}$$

we get fundamental / normal modes

$$u_{m,n}(x,y,t) = X_m(x) Y_n(y) T_{m,n}(t)$$
$$= \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \left(B_{mn} \cos(c k_{m,n} t) + B_{mn}^* \sin(c k_{m,n} t) \right)$$

By superposition principle:

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{m,n}(x,y,t) \text{ solves 2D WEQ.}$$

<u>what about initial conditions?</u>

$$u(x,y,0) = \sum_{n,m=1}^{\infty} B_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y = f(x,y) \quad (1)$$

$$u_t(x,y,0) = \sum_{n,m=1}^{\infty} B_{mn}^* c k_{m,n} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y = g(x,y)$$

(1) is 2D Fourier series of $f(x,y)$. Coefficients can be obtained noting that the functions: $\mathbb{R}^2 \to \mathbb{R}$:

$$\left\{ \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \right\}_{m,n=1}^{\infty}$$ form an orthogonal family.

with respect to inner product:

$$(u,v) = \int_0^a dx \int_0^b dy \; u(x,y) \cdot v(x,y)$$

that is:

$$\left(\sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y, \sin\frac{m'\pi}{a}x \sin\frac{n'\pi}{b}y \right) = 0$$

if $(m,n) \neq (m',n')$

and:

$$\left(\sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y, \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \right) = \left(\sin\frac{m\pi}{a}x, \sin\frac{m\pi}{a}x\right)\left(\sin\frac{n\pi}{b}y, \sin\frac{n\pi}{b}y\right) = \left(\frac{a}{2}\right)\left(\frac{b}{2}\right) = \boxed{\frac{ab}{4}}$$

Thus:

$$\boxed{B_{m,n} = \frac{\left(f(x,y), \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \right)}{\left(\sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y, \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \right)}}$$

in the same way

$$\boxed{B^*_{m,n} \, c \, k_{m,n} = \frac{\left(g(x,y), \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \right)}{\left(\sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y, \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \right)}}$$

Solution of the 2D Heat Eq

Using sep of vars the derivation is similar, what changes is the nature of the time dependent part of the solution, it is exp. decaying instead of periodic.

$$\begin{cases} u_t = c^2(u_{xx} + u_{yy}) & 0<x<a,\ 0<y<b,\ t>0 \\ u(0,y,t) = u(a,y,t) = 0 & 0<y<b \\ u(x,0,t) = u(x,b,t) = 0 & 0<x<a \\ u(x,y,0) = f(x,y) \end{cases}$$

We get: $X_m(x) = \sin \frac{m\pi}{a} x$,

$Y_n(y) = \sin \frac{n\pi}{b} y$,

$$T_{m,n}' + \Big(\underbrace{\left(\tfrac{m\pi}{a}\right)^2}_{\mu_m} + \underbrace{\left(\tfrac{n\pi}{b}\right)^2}_{\nu_n}\Big) c^2 T_{m,n} = 0$$

$$T_{m,n}(t) = B_{m,n} \exp\left(-(\mu_m^2 + \nu_n^2) c^2 t\right)$$

$$\Rightarrow u_{m,n}(x,t) = X_m(x) Y_n(y) T_{m,n}(t)$$
$$= B_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \exp\left(-(\mu_m^2 + \nu_n^2) c^2 t\right)$$

By superposition principle:

$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{m,n}(x,y,t)$ solves 2D Heat Eq as well

What about initial conditions?

$$f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

where

$B_{m,n}$ = coefficient in 2D Fourier series of $f(x,y)$ namely:

$$= \frac{\left(f(x,y),\ \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y\right)}{\left(\sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y,\ \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y\right)}$$

§ 3.8 Laplace Equation in rectangular coordinates

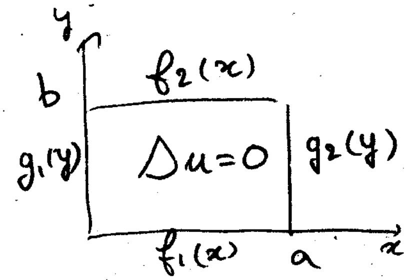
recall heat eq. in 2D: $u_t = c^2(u_{xx} + u_{yy}) = c^2 \underbrace{\Delta u}_{\text{Laplacian}}$

Steady state $u_t = 0 \Rightarrow \underline{\Delta u = 0}$ Laplace equation

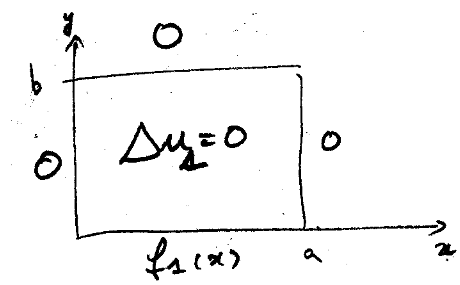
Solving Laplace equation with separation of variables

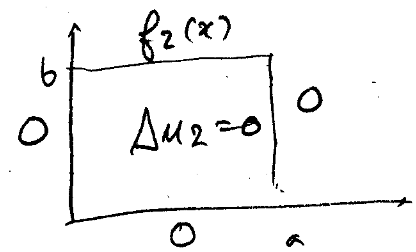
The full blown problem is:

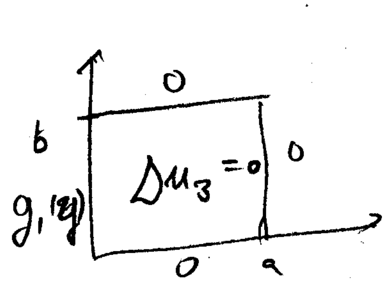
$$(*) \quad \begin{cases} \Delta u = 0 \\ u(x,0) = f_1(x) \\ u(x,b) = f_2(x) \\ u(0,y) = g_1(y) \\ u(a,y) = g_2(y) \end{cases}$$

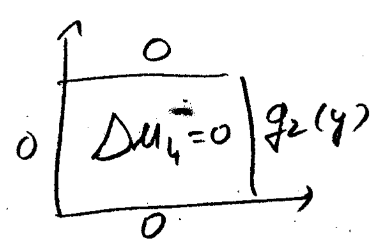


Simplify problem by using linearity: solve for u_1, u_2, u_3, u_4









by linearity the sol to $(*)$ is $u = u_1 + u_2 + u_3 + u_4$

Let us solve for u_1 (cf. example 3.8.1 for u_2)
using sep of variables:

$$u_1(x,y) = X(x)\,Y(y)$$

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = k \text{ constant.}$$

We get:

$$(PX)\begin{cases} X'' + kX = 0 \\ X(0) = X(a) = 0 \end{cases} \qquad (PY)\begin{cases} Y'' - kY = 0 \\ Y(b) = 0 \end{cases}$$

(PX): $k \leq 0$ leads to $X(x) = 0$ (trivial solution)

let $k = +\mu^2 \Rightarrow X(x) = A\cos\mu x + B\sin\mu x$

$$X(0) = A = 0$$

$$X(a) = B\sin\mu a = 0$$

$$\Rightarrow \mu = \mu_n = \frac{n\pi}{a}, \quad n = 1, 2, \dots$$

$$\Rightarrow \boxed{X_n(x) = \sin\frac{n\pi}{a}x}, \quad n = 1, 2 \dots$$

(PY): $Y_n(y) = A_n'\cosh\mu_n y + B_n'\sinh\mu_n y$

B.C.: $Y_n(b) = 0 \Rightarrow A_n'\cosh\mu_n b + B_n'\sinh\mu_n b = 0$

$$\Rightarrow \frac{\sinh\mu_n b}{\cosh\mu_n b} = \tanh\mu_n b = -\frac{A_n'}{B_n'}$$

$$\Rightarrow A_n' = A_n\sinh\mu_n b = A_n\sinh\frac{n\pi}{a}b$$

$$B_n' = -A_n\cosh\mu_n b = -A_n\cosh\frac{n\pi}{a}b$$

$$\Rightarrow \boxed{Y_n(y) = A_n\left[\sinh\mu_n b\cosh\mu_n y - \cosh\mu_n b\sinh\mu_n y\right] = A_n\sinh(\mu_n(b-y))}$$

Recall: $\sinh(a+b) = \sinh(a)\cosh b + \sinh b\cosh a$

Thus $u_n(x,y) = X_n(x)\,Y_n(y) = A_n\sin\frac{n\pi}{a}x\,\sinh(\mu_n(b-y))$

solves PDE by construction, and so does

$$\boxed{u_1(x,y) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi}{a}x\,\sinh(\mu_n(b-y))}$$

Now use boundary condition :

$$u_1(0,x) = f_1(x)$$

$$= \sum_{n=1}^{\infty} \underbrace{A_n \sinh \frac{n\pi}{a} b}_{\text{Sine coeff of } f_1(x)} \sin \frac{n\pi}{a} x$$

$$\Rightarrow \boxed{A_n = \frac{2}{a} \frac{1}{\sinh \frac{n\pi}{a} b} \int_0^a f_1(x) \sin \frac{n\pi}{a} x \, dx}$$

We can do this similarly with the other B.C.

The solution to the problem w/ arbitrary Dirichlet B.C. is:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} (b-y)$$
$$+ B_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y$$
$$+ C_n \sinh \frac{n\pi}{b} (a-x) \sin \frac{n\pi}{b} y$$
$$+ D_n \sinh \frac{n\pi}{b} x \sin \frac{n\pi}{b} y$$

where $A_n \sinh \frac{n\pi}{a} b$ = Sine series coeff of $f_1(x)$

$B_n \sinh \frac{n\pi}{a} b$ = ———— $f_2(x)$

$C_n \sinh \frac{n\pi a}{b}$ = ———— $g_1(y)$

$D_n \sinh \frac{n\pi a}{b}$ = ———— $g_2(y)$.