§ 3.4 D'Alembert's method (purely 1D)

Recall 1DWEQ
$$\begin{cases} u_{tt} = c^2 u_{xx} & , \ 0<x<L, t>0 \\ u(0,t) = u(L,t) = 0 & , \ t>0 \\ u(x,0) = f(x) & , \ 0<x<L \\ u_t(x,0) = g(x) & , \ 0<x<L \end{cases}$$

D'Alembert's solution:
$$\boxed{u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds} \quad (*)$$

where $f^* \equiv 2L$ per ODD ext. of f

$g^* \equiv$ ——————— g

But we drop $*$ notation below to keep notation simple.

Consistency check:

Can $u_n(x,t) = \sin \frac{n\pi}{L}x \cos \frac{cn\pi}{L}t$ be written as $(*)$?

$$\left.\begin{aligned} \sin(a+b) &= \sin a \cos b + \cos a \sin b \\ \sin(a-b) &= \sin a \cos b - \cos a \sin b \end{aligned}\right\} \Rightarrow \sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

$$\Rightarrow \quad u_n(x,t) = \frac{1}{2}\left[\sin \frac{n\pi}{L}(x+ct) + \sin\frac{n\pi}{L}(x-ct)\right]$$

Does $(*)$ solve 1DWEQ?

To compute derivatives of $(*)$ we need the following formula from calc III:

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} f(s;t)\,ds\right] = b'(t) f(b(t),t) - a'(t) f(a(t),t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(s,t)\,ds$$

Hence:

$$u_t = \frac{c}{2}\left[-f'(x-ct)+f'(x+ct)\right] + \frac{1}{2}\left[g(x+ct)+g(x-ct)\right]$$

$$u_{tt} = \frac{c^2}{2}\left[f''(x-ct)+f''(x+ct)\right] + \frac{c}{2}\left[g'(x+ct)-g'(x-ct)\right]$$

$$u_x = \frac{1}{2}\left[f'(x-ct)+f'(x+ct)\right] + \frac{1}{2c}\left[g(x+ct)-g(x-ct)\right]$$

$$u_{xx} = \frac{1}{2}\left[f''(x-ct)+f''(x+ct\right] + \frac{1}{2c}\left[g'(x+ct)-g'(x-ct)\right]$$

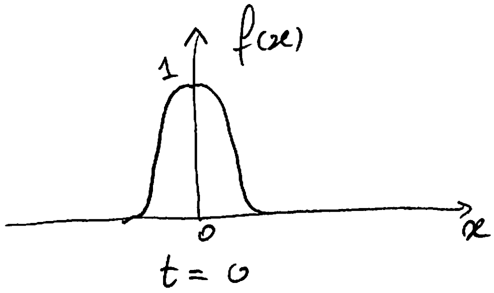
therefore we do have $u_{tt} = c^2 u_{xx}$.

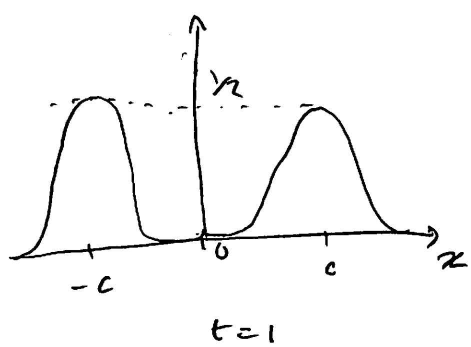
Are we done? No we need to check B.C.

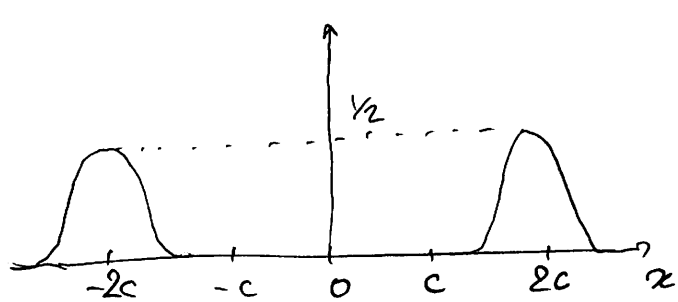
$$u(0,t) = \frac{1}{2}\left[f^*(-ct)+\underbrace{f^*(ct)}_{\text{odd ext.}}\right] + \frac{1}{2c}\int_{-ct}^{ct} \underbrace{g^*(s)}_{\text{odd}}\,ds = 0$$

Physical interpretation

Assume $g(x)=0 \Rightarrow u(x,t) = \frac{1}{2}\left[f^*(x-ct)+f^*(x+ct)\right]$







What happen in the general case (i.e. $g(x) \neq 0$)

let

$$G(x) = \int_a^x g^*(z)\, dz$$

$$G(x+2L) - G(x) = \int_x^{x+2L} g^*(z)\, dz \overset{g^* \text{ is } 2L\text{-per}}{=} \int_{-L}^{L} g^*(z)\, dz \overset{g^* \text{ is odd}}{=} 0$$

$\Rightarrow$ G is $2L$-periodic

$$\Rightarrow u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right] + \frac{1}{2c}\left[G(x+ct) - G(x-ct)\right]$$

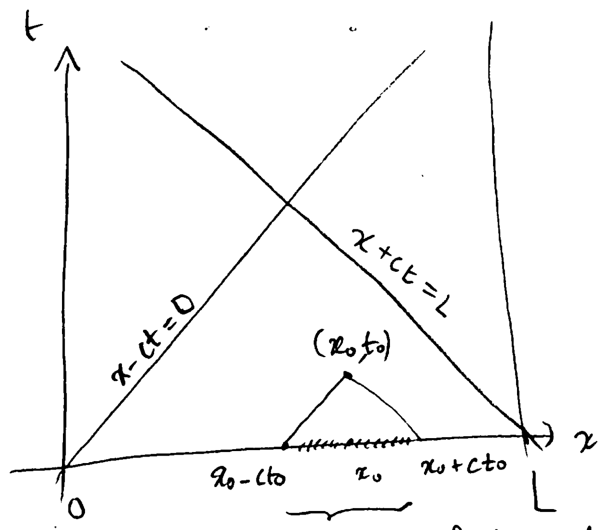
$$= \underbrace{\frac{1}{2}\left[f^*(x-ct) - \frac{1}{c} G(x-ct)\right]}_{\text{right propa. term}} + \underbrace{\frac{1}{2}\left[f^*(x+ct) + \frac{1}{c} G(x+ct)\right]}_{\text{left propa term}}$$

Characteristics

Characteristics = curves where sol is const.

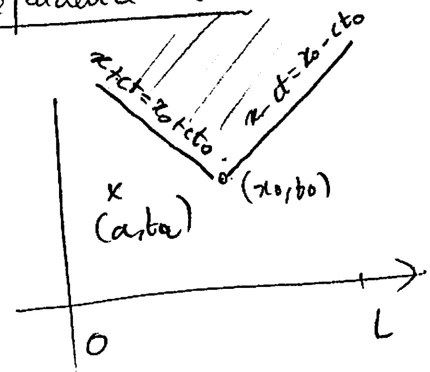
$$x - ct = x_0 - ct_0$$
$$x + ct = x_0 + ct_0$$

\ characteristics have slope $\pm \frac{1}{c}$ in (x,t) plane



interval of dependence: value of $u(x_0,t_0)$ can only be influenced by sol at this interval

Dependence cone:



because waves propagate at a finite speed, only observers in cone in xt plane can see disturbance at (x_0,t_0).

Observer at (a,t_a) will not see disturbance!