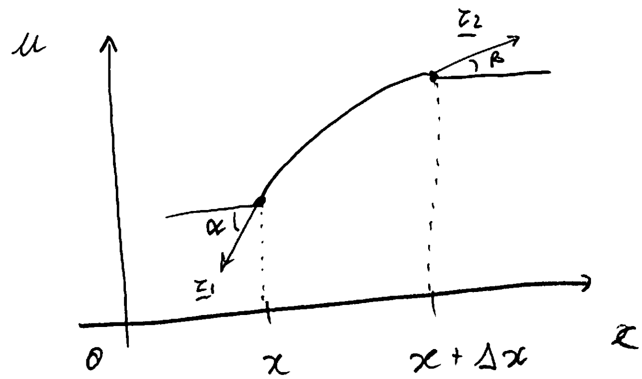
§ 3.2 Derivation of 1D Wave equation (only as motivation for math) (you don't need to know this)



u = displacement of string.

This is a snapshot of a small piece of string located in pos $\in [x, x+\Delta x]$

$\underline{\tau}_1$ = tension force exerted by rest of string on this little piece at x
$\underline{\tau}_2$ = " " " " " " " " " " at $x+\Delta x$

<u>Assumption</u>: • length of string changes very little w/ small displacements

$\Rightarrow \quad |\underline{\tau}_1| = |\underline{\tau}_2| = \tau$ i.e. constant tension force over whole string.

• only look at vertical component of displacement (horizontal movement is negligible)

<u>Idea</u>: use Newton's second law:

$$\boxed{\sum F = ma}$$

let ρ = mass density of string $\Rightarrow$ mass of little piece $[x, x+\Delta x]$ is $\rho \Delta x$

- <u>mass of $[x+\Delta x]$ × acceleration of $[x+\Delta x]$</u> = $\rho \Delta x \dfrac{\partial^2 u}{\partial t^2}(x,t)$

- <u>sum of forces</u>: $\tau \sin\beta - \tau \sin\alpha$ (we are only looking at vertical component. the horizontal displacement is assumed to be zero)

Hence applying Newton's second law:

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = \tau (\sin\beta - \sin\alpha) \approx \tau (\tan\beta - \tan\alpha)$$
$$= \tau \left(\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x, t) \right)$$

Letting $\Delta x \to 0$:

$$\frac{\rho}{\tau} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} = \text{WEQ}.$$

where $c^2 = \frac{\rho}{\tau}$ = speed squared

(Note: $\lim_{\Delta x \to 0} \dfrac{\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)}{\Delta x} = \dfrac{\partial^2 u}{\partial x^2}(x,t)$)

§ 3.3 Solution of 1D WEQ using separation of variables

1DWEQ:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L \\ u(0,t) = 0; \ u(L,t) = 0 & t > 0 \quad \text{(B.C.)} \\ u(x,0) = f(x) & \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & \Big\} \ 0 < x < L \quad \text{(I.C.)} \end{cases}$$

$\boxed{\text{If you understand the spirit of this section, you understand 90\% of class}}$

We will use (again and again) the very useful method of:

<u>separation of variables</u>.

<u>Idea</u>: from German "attempt", "approach"

- Use ansatz for solution:

$$\boxed{u(x,t) = X(x)\, T(t)}$$

(hence the name ... ansatz depends on problem, of course)

- plug ansatz into PDE
- interpret and solve a sys of ODEs
- go back to original problem

With this ansatz:

$$\frac{\partial^2 u}{\partial x^2} = X'' T \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = X T''$$

So WEQ becomes:

$$X T'' = c^2 X'' T$$

$$\Leftrightarrow \quad \underbrace{\frac{T''}{c^2 T}}_{f(t)} = \underbrace{\frac{X''}{X}}_{g(x)} \qquad \text{for all } x, t.$$

The only way we can have equality for all x, t is if

$$f(t) = g(x) = \text{constant } k$$

Thus we get 2 ODEs: (coupled by the k)

$$\frac{X''}{X} = k \quad \text{and} \quad \frac{T''}{c^2T} = k$$

$$\begin{cases} X'' = kX & (1) \\ T'' = kc^2T & (2) \end{cases}$$ + B.C. given by B.C. of WEQ

▶ Let us solve (1) first: (it looks easier)

$$\begin{cases} X'' = kX \\ X(0) = X(L) = 0 \end{cases}$$

B.C. $u(0,t) = X(0)T(t) = 0$
$\Leftrightarrow X(0) = 0$
$u(L,t) = X(L)T(t) = 0$
$\Leftrightarrow X(L) = 0$

Solution of this ODE depends on sign of k

See Appendix A.2 for refresh

• case $k = \mu^2 > 0$

$X(x) = a\cosh\mu x + b\sinh\mu x$

$X(0) = a = 0$

$X(L) = b\underbrace{\sinh\mu L}_{\neq 0} = 0 \Rightarrow b = 0$

$\Rightarrow X(x) = 0$ trivial solution, so this case is not interesting case $k=0$:

$$\left.\begin{aligned} X(x) &= ax+b \\ X(0) &= b = 0 \\ X(L) &= aL = 0 \Rightarrow a=0 \end{aligned}\right\}$$

again trivial sol $X(x)=0$

case $k=-\mu^2<0$

$X(x) = a\cos\mu x + b\sin\mu x$

$X(0) = a = 0$

$X(L) = b\sin\mu L = 0 \Rightarrow \mu L = n\pi,\ n\in\mathbb{Z}, n\neq 0$

$\Rightarrow \boxed{\mu_n = \frac{n\pi}{L}}$

can restrict ourselves to $n\geq 1$ since $X_n = -X_{-n}$

$\Rightarrow \boxed{X_n(x) = \sin\frac{n\pi}{L}x}$ where we took $b=1$ to make all subsequent expressions easier.

▶ Continue w/ eq (2)

$T_n'' = -\left(c\frac{n\pi}{L}\right)^2 T_n \Rightarrow$

λ_n in book

$$\boxed{T_n(t) = b_n\cos\frac{cn\pi}{L}t + b_n^*\sin\frac{cn\pi}{L}t}$$

So sol to 1D WE Q w/ BC $u_n(0,t)=u_n(L,t)=0$ is:

$$\boxed{u_n(x,t) = X_n(x)T_n(t) = \sin\frac{n\pi}{L}x\left(b_n\cos\frac{cn\pi}{L}t + b_n^*\sin\frac{cn\pi}{L}t\right)}$$

$u_n(x,t)$ = normal modes of wave equation.

WEQ = linear and homogeneous, so we can use the superposition principle

Are we done? Not yet, we still need to take care of I.C. !

We look for solution of the form:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x \left(b_n \cos\frac{n\pi}{L}ct + b_n^* \sin\frac{n\pi}{L}ct \right)$$

* Init cond $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \overset{\text{I.C.}}{=} f(x)$$

thus b_n are coeff of 2L-per ODD expansion of $f(x)$ (sine series)

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx} \quad n \geq 1$$

* Init cond $u_t(x,0) = g(x)$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \left(\sin\frac{n\pi}{L}x\right)\frac{n\pi c}{L}\left(-b_n \sin\frac{n\pi}{L}ct + b_n^* \cos\frac{n\pi}{L}ct\right)$$

$$\Rightarrow \underset{\substack{\| \\ g(x)}}{u_t(x,0)} = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n^* \sin\frac{n\pi}{L}x$$

$\Rightarrow$ $\frac{n\pi c}{L} b_n^*$ = sine series coeff of $g(x)$

$$= \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$\rightarrow \quad \boxed{b_n^* = \underbrace{\frac{L}{n\pi c} \times \frac{2}{L}}_{2/(n\pi c)} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx}$$

Summary: The sol to 1D WEQ

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x) & 0 < x < L \\ u_t(x,0) = g(x) & 0 < x < L \end{cases}$$

is:
$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[b_n \cos \frac{n\pi}{L} c t + b_n^* \sin \frac{n\pi}{L} c t \right]$$

where
$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

$$b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

Example (see also book example, this is ~3.3.5)

1DWEQ with $c=4$, $f(x) = \begin{cases} x & 0 \le x \le \frac{1}{2} \\ (1-x) & \frac{1}{2} \le x \le 1 \end{cases}$ $(L=1)$

$$g(x) = 0 \Rightarrow b_n^* = 0$$

We need to compute sine series of $f(x)$:

$$b_n = 2\int_0^1 f(x)\sin n\pi x\,dx = 2\int_0^{\frac{1}{2}} x\sin n\pi x\,dx + 2\int_{\frac{1}{2}}^1 (1-x)\sin n\pi x\,dx$$

$$= \cancel{-\frac{2}{n\pi}x\cos n\pi x\Big|_0^{\frac{1}{2}}} + \frac{2}{n\pi}\int_0^{\frac{1}{2}}\cos n\pi x\,dx$$

$$-\frac{2}{n\pi}(1-x)\cos n\pi x\Big|_{\frac{1}{2}}^1 - \frac{2}{n\pi}\int_{\frac{1}{2}}^1 \cos n\pi x\,dx$$

$$= \frac{2}{(n\pi)^2}\left[\sin n\pi x\Big|_0^{\frac{1}{2}} - \sin n\pi x\Big|_{\frac{1}{2}}^1\right] = \boxed{\frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}}$$

$$\Rightarrow u(x,t) = \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\left((2k+1)\pi x\right)\cos\left(4(2k+1)\pi t\right)$$

See Matlab implementation.

$n=$	$\sin n\pi/2$
0	0
1	1
2	0
3	-1