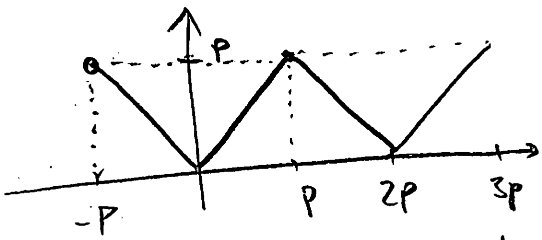
Example $f(x)=|x|$ if $-p\le x\le p$, $2p$- periodic



method 1: Compute integrals to get a_0, b_i (see Example 2.3.1)

method 2: Use Fourier series for the 2π-per. function:

$$g(y)=|y|, \text{ if } -\pi\le y<\pi$$

$$g(y)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left((-1)^n-1\right)\cos ny$$

(we computed this in an earlier example)

By stretching and rescaling g:

$$g\left(\frac{\pi}{p}x\right)=\frac{\pi}{p}|x| \quad \text{if } -\pi\le\frac{\pi}{p}x\le\pi$$

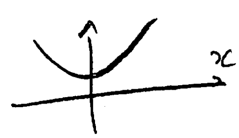
$$=\frac{\pi}{p}|x| \quad \text{if } -p< x\le p$$

Hence:

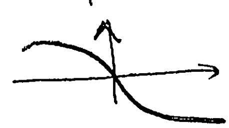
$$\boxed{f(x)=\frac{p}{\pi}g\left(\frac{\pi}{p}x\right)=\frac{p}{2}+\sum_{n=1}^{\infty}\frac{2p}{\pi^2 n^2}\left((-1)^n-1\right)\cos\frac{n\pi}{p}x}$$

Even and odd functions

Recall: f is even $\iff f(-x)=f(x)$ for all x



f is odd $\iff f(-x)=-f(x)$ for all x



$$\int_{-p}^{p}f(x)\,dx=\begin{cases}0 & \text{if } f \text{ is ODD}\\ 2\int_0^p f(x)\,dx & \text{—— EVEN}\end{cases}$$

Also:

even $\times$ even	=	even
even $\times$ odd	=	odd
odd $\times$ odd	=	even

Implications of parity of a function for Fourier series

- If f is $2p$-per. EVEN fun:

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{\underbrace{f(x)}_{\text{even}}\ \underbrace{\sin\frac{n\pi}{p}x}_{\text{odd}}}_{\text{odd}}\,dx = 0$$

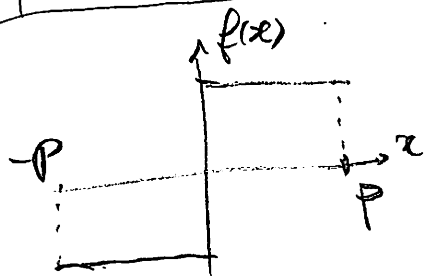
$$\Leftrightarrow \boxed{\underset{\uparrow\ \text{even}}{f(x)} = \underset{\uparrow\ \text{even}}{a_0} + \sum_{n=1}^{\infty} a_n \underbrace{\cos\frac{n\pi x}{p}}_{\text{even}}}$$

- If f is $2p$-periodic ODD fun

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\frac{n\pi}{p}x}_{\text{odd}}\,dx = 0$$

$$\Leftrightarrow \boxed{\underset{\uparrow\ \text{odd}}{f(x)} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x}$$

cf. Examples 2.3.4, 2.3.5 and problem 2.3.1



f = odd function $\Rightarrow a_n = 0$

$$b_n = \frac{2}{p}\int_0^p \sin\frac{n\pi}{p}x\,dx$$

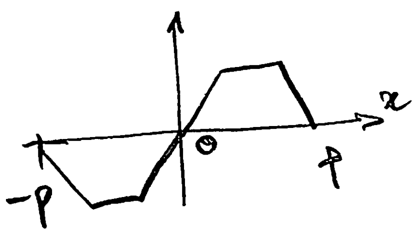
$$= -\frac{2}{p}\frac{p}{n\pi}\cos\frac{n\pi}{p}x\Big|_0^p$$

$$= -\frac{2}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

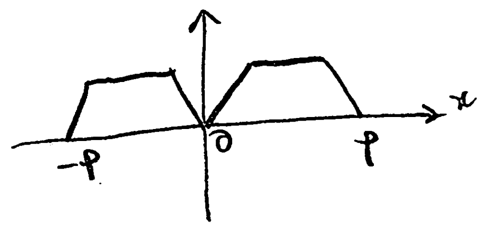
$$\Rightarrow \boxed{f(x) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin\frac{(2k+1)}{p}\pi x}$$

Half Range Expansions: the cosine and sine series

<u>Idea</u>: A function f defined on $[0,p]$ can be extended to a $2p$-periodic function as follows:



<u>odd</u> extension



<u>even</u> extension.

<u>Cosine</u> series ≡ Fourier series of <u>even</u> extension

<u>sine</u> series ≡ " " " <u>odd</u> "

<u>Theorem (Half range expansions)</u> Let $f(x)$ be a piecewise smooth function defined on $[0,p]$.

<u>Cosine series expansion of $f(x)$</u> is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x \quad \text{for } 0 \le x \le p$$

(even by construction)

$$\text{where } a_n = \frac{\left(f, \cos \frac{n\pi}{p} x\right)}{\left(\cos \frac{n\pi}{p} x, \cos \frac{n\pi}{p} x\right)} = \underbrace{\frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x \, dx}$$

same formula as Fourier series of f, but pretending f is EVEN: $f(-x) = f(x)$

$$\Rightarrow \boxed{\begin{aligned} a_0 &= \frac{1}{2p}\,2\int_0^p f(x)\,dx \\ a_n &= \frac{1}{p}\,2\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx \end{aligned}}$$

⚠ it is easier to rederive these formulae than remembering them.

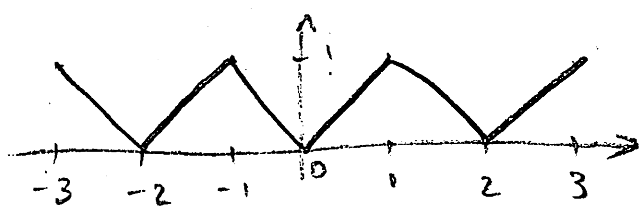
<u>Sine series expansion</u>: Similarly:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \quad \text{for } 0 \le x \le p$$

where
$$\boxed{b_n = \frac{\left(f, \sin\frac{n\pi}{p}x\right)}{\left(\sin\frac{n\pi}{p}x, \sin\frac{n\pi}{p}x\right)} = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx}$$

<u>Example</u> Consider $f(x) = x, \quad 0 \le x \le 1.$

<u>Even (2-per) extension</u>

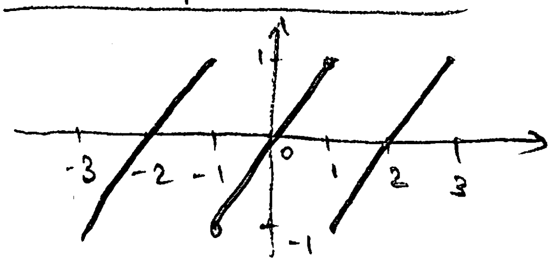


$$\boxed{a_0 = \frac{2}{2}\int_0^1 x\,dx = \frac{1}{2}}$$

$$\begin{aligned} a_n &= \frac{2}{1}\int_0^1 x\cos n\pi x\,dx \\ &= 2x\frac{\sin n\pi x}{n\pi}\Big|_0^1 - \frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx \\ &= \frac{2}{(n\pi)^2}\cos n\pi x\Big|_0^1 = \frac{2}{(n\pi)^2}\left((-1)^n - 1\right) \\ &= \begin{cases} -\dfrac{4}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \end{aligned}$$

$$\Rightarrow \boxed{x = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)\pi x}$$

<u>odd (2-per) extension</u>



$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 x\sin n\pi x\,dx \\ &= -2x\frac{\cos n\pi x}{n\pi}\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos n\pi x\,dx \\ &= -\frac{2(-1)^n}{n\pi} + \cancel{\frac{2}{(n\pi)^2}\sin n\pi x\Big|_0^1} \\ &\qquad\qquad\quad = 0 \end{aligned}$$

$$\Rightarrow \boxed{x = \frac{-2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x}$$