§ 2.1 Periodic Functions

Def: A function f is periodic with period T iff

$$f(x+T) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

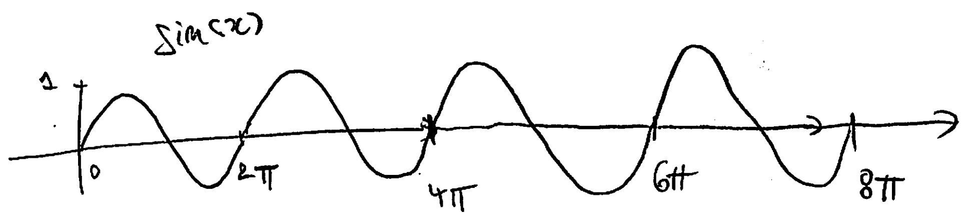
(a.k.a. T-periodic function)

Examples: $\sin(x)$ is 2π-periodic

$\sin(nx)$ is $\frac{2\pi}{n}$-periodic

Note: $f(x) = f(x+T) = f(x+2T) = \cdots = f(x+nT)$

f is T-periodic $\Rightarrow$ f is nT-periodic for $n \geq 1$



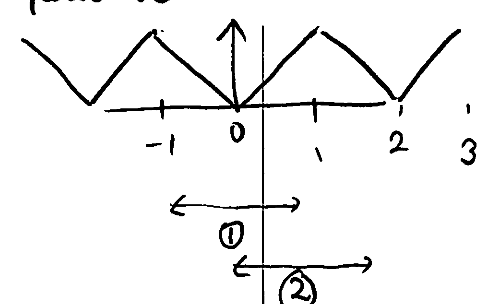
$\sin(x)$ is also 4π-periodic, $2n\pi$-periodic for $n \geq 1$.

Note: A T-periodic can be defined in many different but equivalent ways (it depends on which interval is used initially for "copy paste")

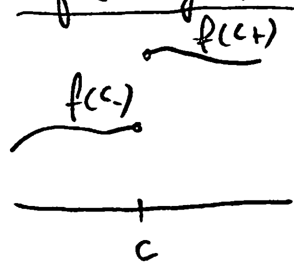
① $f(x) = |x|$ for $-1 \leq x \leq 1$ and 2-periodic

$\Leftrightarrow$

② $f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 2-x & \text{for } 1 < x < 2 \end{cases}$



Def (right, left limit)



$$f(c+) = \lim_{x \to c+} f(x) = \lim_{\substack{h \to 0 \\ h > 0}} f(c+h)$$

= limit from the <u>right</u>

$$f(c-) = \lim_{x \to c-} f(x) = \lim_{\substack{h \to 0 \\ h > 0}} f(c-h)$$

= limit from the <u>left</u>.

Def (continuity) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say f is a <u>continuous</u> function when:

$$\lim_{x \to c} f(x) = f(c) \qquad \text{for all } c \in \mathbb{R}$$

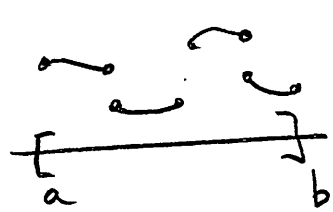
or equivalently:

$$\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) = f(c)$$

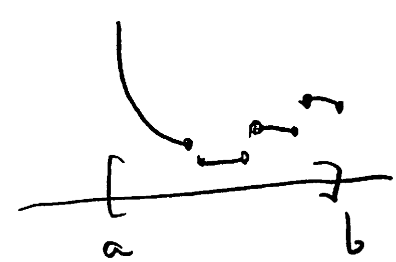
Def (piecewise continuous) A function f is <u>piecewise continuous</u> on an interval $[a,b]$ if:

i) $f(a+)$ and $f(b-)$ exist

ii) f is defined and continuous on $[a,b]$ except at a <u>finite number</u> of points in (a,b), where left & right limits exist but are different.



pws cont.

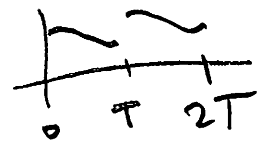


Here $f(a+)$ does not exist so the f is <u>not</u> pws cont.

For periodic functions:

f is pcws cont on a period $\Leftrightarrow$ f is pcws cont on $\mathbb{R}$
(e.g. $[0,T]$)

however
f is cont on a period $\Rightarrow$ f is pcws cont on $\mathbb{R}$

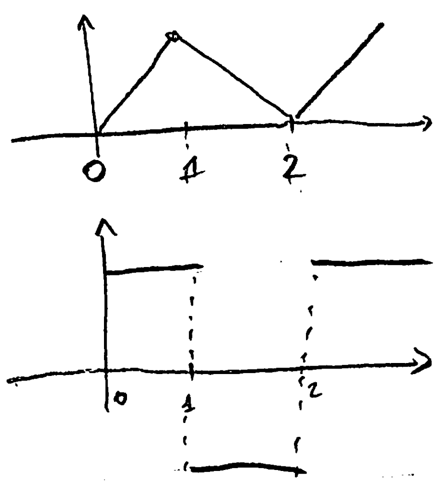


(Simply because values at start and end of period may not match)

For a per. fun to be cont on $\mathbb{R}$ we need:
i) f cont on $[0,T]$
ii) $f(0+) = f(T-)$

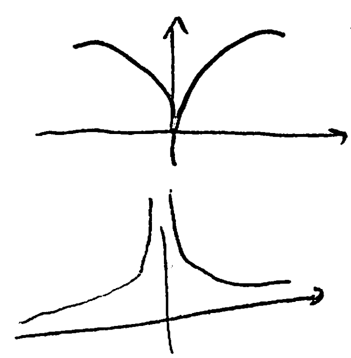
Def (pcws smooth) f is pcws smooth on $[a,b]$ iff f and f' are pcws cont. on $[a,b]$.

f



f'

pcws smooth



$\sqrt{|x|}$ = pcws cont but not pcws smooth

$\frac{1}{2\sqrt{|x|}}$

Integral over a period

Let f be a T-periodic function then:

$$\int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx \,, \text{ for all } a \in \mathbb{R}.$$

Proof Assume f is continuous (proof extends in general)

$$F(a) = \int_a^{a+T} f(x)\,dx$$

$$F'(a) = f(a+T) - f(a) = 0$$

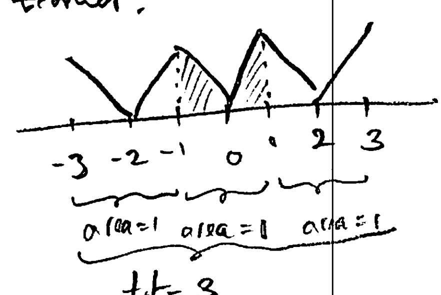
$\Rightarrow F(a) = \text{const}.$ QED. (Quod est demonstratum)

Example: Let $f(x) = |x|$ for $-1 \le x \le 1$ and 2-periodic.

(a) $\displaystyle\int_{-1}^{1} |x|\,dx = 2\int_0^1 x\,dx = 1$

(b) $\displaystyle\int_{-N}^{N} f(x)\,dx = \underbrace{\int_{-N}^{-N+2} f(x)\,dx + \int_{-N+2}^{-N+4} f(x)\,dx + \dots + \int_{N-2}^{N} f(x)\,dx}_{N \text{ times.}}$

$= N$



$L^2[a,b]$ inner product:

Let u, v be real valued functions defined on $[a,b]$.

The inner product of u an v is:

$$\boxed{(u,v) = \int_a^b u(x)\, v(x)\, dx} \quad (* \text{ important } *)$$

How do we recognize this is an inner prod? [optional]

Because it is a positive symmetric bilinear form:

i) $(u,v) = \int_a^b u(x) v(x)\, dx = \int_a^b v(x) u(x)\, dx = (v,u)$

(symm)

ii) $(u+v, w) = \int_a^b (u+v)(x)\, w(x)\, dx$

$= \int_a^b u(x) w(x)\, dx + \int_a^b v(x) w(x)$

$= (u,w) + (v,w)$

$(\alpha u, v) = \int_a^b \alpha u(x) v(x)\, dx = \alpha \int_a^b u(x) v(x)\, dx$

$= \alpha\, (u,v)$

(linear)

iii) $(u,u) = \int_a^b u^2(x)\, dx \geq 0$

$(u,u) = 0 \Leftrightarrow \int_a^b u^2(x)\, dx = 0 \Leftrightarrow u(x) = 0$

(positive)

Def ($\perp$ of functions)

Two functions u and v are said to be orthogonal if

$$(u, v) = 0 \quad \text{(compare to notion in } \mathbb{R}^n\text{)}$$

An *important* family of orthogonal functions on $[-\pi, \pi]$ is the *trigonometric system*

$$1, \ \cos x, \ \cos 2x, \ \cos 3x, \ \dots, \cos mx, \ \dots$$
$$\sin x, \ \sin 2x, \ \sin 3x, \ \dots, \sin mx, \ \dots$$

Check: We need to verify that if we take inner prod of any function in family with another (different) function from family we get zero i.e:

$$(\cos mx, \cos nx) = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n > 0 \\ 2\pi & \text{if } m = n = 0 \end{cases} \quad (1)$$

$$(\sin mx, \sin nx) = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \quad (2)$$

$$(\cos mx, \sin nx) = 0 \quad \text{for all } n, m \quad (3)$$

We evaluate these integrals using trig identities.

- Case ①:

$$\begin{cases} \cos(m+n)x = \cos mx \cos nx - \sin mx \sin nx \\ \cos(m-n)x = \cos mx \cos nx + \sin mx \sin nx \end{cases}$$

$$\Rightarrow \quad \cos mx \cos nx = \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]$$

$$\Rightarrow \quad (\cos mx, \cos nx) = \frac{1}{2}\int_{-\pi}^{\pi} (\cos(m+n)x + \cos(m-n)x)\,dx$$

$$\overset{n \neq m}{=} \frac{1}{2(m+n)} \sin(m+n)x + \frac{1}{2(m-n)} \sin(m-n)x \Big|_{-\pi}^{\pi}$$

$$= 0$$

$$(\cos nx, \cos nx) = \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos(2nx)\,dx = \frac{1}{2} \times 2\pi = \pi$$

$$(1, 1) = \int_{-\pi}^{\pi} 1\,dx = 2\pi$$

- Case ②:

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\Rightarrow \quad (\sin mx, \sin nx) = \frac{1}{2}\int_{-\pi}^{\pi} (\cos(m-n)x - \cos(m+n)x)\,dx$$

$$\overset{n \neq m}{=} \frac{1}{2}\left(\frac{1}{m-n}\sin(m-n)x - \frac{1}{m+n}\sin(m+n)x\right)\Big|_{-\pi}^{\pi}$$

$$= 0$$

$$(\sin nx, \sin nx) = \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2nx)\,dx = \frac{1}{2}\, 2\pi = \pi$$

Case B :

$$\sin(m+n)x = \sin mx \cos nx - \cos mx \sin nx$$
$$\sin(m-n)x = \sin mx \cos nx + \cos mx \sin nx$$

$$\Rightarrow \quad \sin mx \cos nx = \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right]$$

$$(\sin mx, \cos nx) = \frac{1}{2}\int_{-\pi}^{\pi} (\sin(m+n)x + \sin(m-n)x)\,dx$$

$$\overset{n\neq m}{=} \frac{1}{2}\left[-\frac{1}{m+n}\cos(m+n)x - \frac{1}{m-n}\cos(m-n)x\right]\Big|_{-\pi}^{\pi}$$

$$= 0$$

$$(\sin nx, \cos nx) = \frac{1}{2}\int_{-\pi}^{\pi} \sin(2nx)\,dx = \frac{-1}{2n}\cos(2nx)\Big|_{-\pi}^{\pi} = 0$$

Fourier Series §2.2

Idea: orthogonal basis in $\mathbb{R}^n$ suggests that any suitable function f on $[-\pi, \pi]$ should be able to be expressed in basis of cos & sin :

$$1, \ \cos x, \ \cos 2x, \ \dots$$
$$\sin x, \ \sin 2x, \ \dots$$

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}$$

a_i, b_i are Fourier coeff of f

How to find such coeff? In the same way we find coefficients in expansion of a vector of $\mathbb{R}^n$ in an $\perp$ basis.

$$\boxed{\begin{aligned} a_0 &= \frac{(f,1)}{(1,1)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{(f,\cos nx)}{(\cos nx,\cos nx)} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n \geq 1 \\ b_n &= \frac{(f,\sin nx)}{(\sin nx,\sin nx)} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n \geq 1 \end{aligned}} \quad (*)$$

proof: (only for a_n $n \geq 1$, everything else similar)

$$(f, \cos mx) = \left(a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \cos mx\right)$$

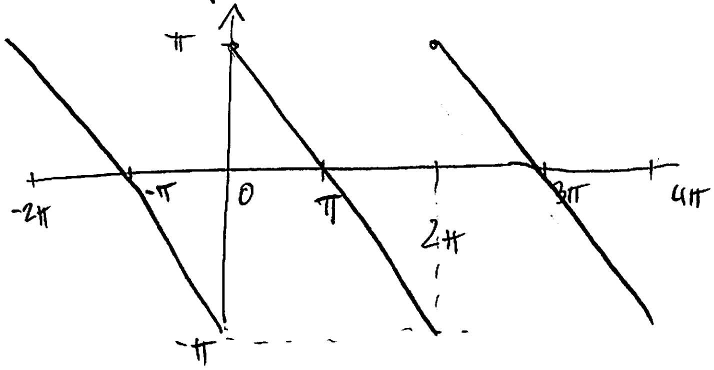
$$= a_m (\cos mx, \cos mx) + a_0 \underbrace{(1, \cos mx)}_{}$$

$$+ \sum_{\substack{n=1 \\ n \neq m}}^{\infty} a_n \underbrace{(\cos nx, \cos mx)}_{=0}$$

$$+ \sum_{n=1}^{\infty} b_n \underbrace{(\sin nx, \cos mx)}_{=0}$$

Note: If f is 2π-periodic, the formulas $(*)$ can be written equivalently on say $[0, 2\pi]$.

Example: $f(x) = \pi - x$ for $0 \le x \le 2\pi$, and 2π-periodic.



$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\cos nx \, dx$$

$$\overset{IBP}{=} \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\Big|_0^{2\pi} + \int_0^{2\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin nx \, dx$$

$$\overset{IBP}{=} \frac{1}{\pi}\left[-(\pi - x)\frac{\cos nx}{n}\Big|_0^{2\pi} - \overset{=0}{\cancel{\int_0^{2\pi}\frac{\cos nx}{n}\,dx}}\right]$$

$$= \frac{1}{\pi}\left(\pi\frac{\cos 2\pi}{n} + \pi\frac{\cos 0}{n}\right) = \frac{2}{n}$$

$$\Rightarrow \boxed{f(x) \text{ "="} \sum_{n=1}^{\infty} \frac{2}{n}\sin nx}$$

equal sign to be explained in a sec.

- Falstad applet
- Gibbs phenomenon

Very strange thing about this approx: $S_N(x) = \sum_{n=1}^{N} \frac{2}{n} \sin nx$ (23)

$\sin n0 = 0 \Rightarrow \boxed{S_N(0) = 0}$

however $f(0) \neq 0$ actually we only have:

$f(0_+) = \pi$

$f(0_-) = -\pi$

$\Rightarrow S_N(0) = \frac{f(0_+) + f(0_-)}{2}$

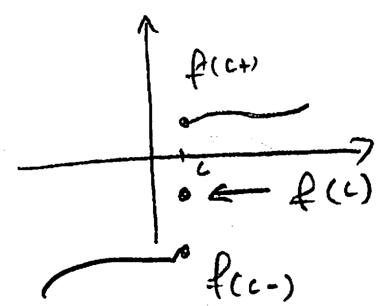
Fourier series representation let f be a 2π-periodic pcws smooth f.

Then for all x we have:

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

So we get pointwise convergence so long as we "hack" functions so that at every point of discontinuity we have:

$$\boxed{f(c) \equiv \frac{f(c_-) + f(c+)}{2}}$$



If we make sure all functions we work with are modified in this way our original expression:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad \underline{\text{works}}.$$