$\mathbb{R}^n$ interlude

Here are some concepts from vector calculus that will be useful to understand the methods we will study

$$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n)^T \mid x_1 \in \mathbb{R}, x_2 \in \mathbb{R} \dots x_n \in \mathbb{R} \}$$

= set of vectors with n real entries.

Example: $\mathbb{R}^2 \equiv$ plane, $\underline{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$

$\mathbb{R}^3$ = 3d space $\underline{x} = \begin{pmatrix} -1 \\ \frac{\pi}{2} \end{pmatrix} \in \mathbb{R}^3$

etc...

The notion of angle and length in $\mathbb{R}^n$ is given by: inner product (a.k.a. dot product, scalar prod.)

Def (inner prod in $\mathbb{R}^n$):

$$(\underline{x}, \underline{y}) = \underline{x}^T \underline{y} = \sum_{i=1}^{n} x_i y_i$$ = sum of prod of entries.

Example: $\underline{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\underline{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$(\underline{x}, \underline{y}) = 2 \times 2 + 2 \times 1 = 6$$

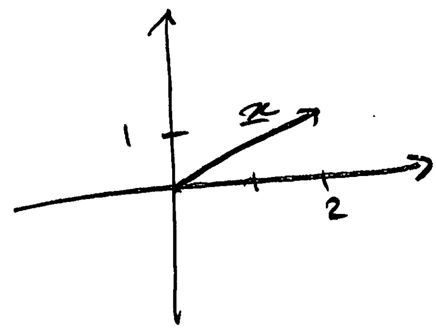
• <u>Length using inner prod</u>:

The length of a vector $\underline{x}$ is denoted $\|\underline{x}\|$ and:

$$\|x\| = (\underline{x}, \underline{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{n} x_i\right)^{\frac{1}{2}}$$

<u>Example</u>:

$$\underline{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \|x\| = \sqrt{4+1} = \sqrt{5}$$

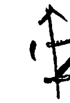


• <u>Angle using inner prod</u>:

$$(\underline{x}, y) = \|x\| \, \|y\| \cos\theta, \quad \theta \equiv \text{angle between } x, y.$$

Gives way of computing angles easily in $\mathbb{R}^n$.

<u>Example</u>: What is angle between $\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:



$45° = \frac{\pi}{4}$

$$\underline{x} \cdot \underline{y} = 1 = \underbrace{\|x\|}_{\sqrt{2}} \, \underbrace{\|y\|}_{1} \cos\theta$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

Def (Orthogonality)

Two vectors are orthogonal ($\perp$) iff

$$(\underline{x}, \underline{y}) = 0$$

* **Theorem (orthogonal basis)**

Let $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ be n $\perp$ vectors $\neq 0$,
(i.e. $(\underline{v}_i, \underline{v}_j) = 0$ if $i \neq j$)

Then any vector $\underline{x}$ in $\mathbb{R}^n$ can be expressed as a linear comb. of these vectors.

$$\underline{x} = \sum_{i=1}^{n} x_i \underline{v}_i = \sum_{i=1}^{n} \frac{(\underline{x}, \underline{v}_i)}{(\underline{v}_i, \underline{v}_i)} \underline{v}_i$$

Proof.
The coeff x_i are determined uniquely since

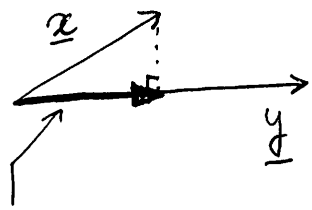
$$(\underline{x}, \underline{v}_j) = \left(\sum_{i=1}^{n} x_i \underline{v}_i, \underline{v}_j\right)$$

$$= \sum_{i=1}^{n} x_i (\underline{v}_i, \underline{v}_j)$$

$$= x_j (\underline{v}_j, \underline{v}_j)$$

$$\Rightarrow \boxed{x_j = \frac{(\underline{x}, \underline{v}_j)}{(\underline{v}_j, \underline{v}_j)}}, \quad j = 1 \ldots n.$$

Projection

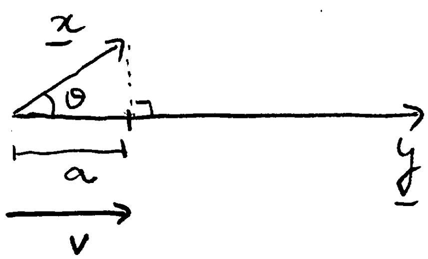


$$\underline{v} = \frac{(\underline{x}, \underline{y})}{(\underline{y}, \underline{y})}\,\underline{y}$$

The projection of a vector $\underline{x}$ onto another vector $\underline{y}$ is: the vector $\underline{v}$

$$\boxed{\underline{v} = \frac{(\underline{x}; \underline{y})}{(\underline{y}, \underline{y})}\,\underline{y}}$$

We can check this using our knowledge of projections from geometry and the notion of angle that is defined by the inner product:



length $a = \|x\| \cos\theta$

$\perp$ proj vector of $\underline{x}$ onto $\underline{y}$ is:

$$\underline{v} = a \times \frac{\underline{y}}{\|\underline{y}\|}$$

← vector of length 1 in the direction 1.

$$= \|x\| \cos\theta \, \frac{\underline{y}}{\|\underline{y}\|}$$

$$= \frac{\|x\|\,\|y\| \cos\theta}{\|y\|^2}\,\underline{y}$$

$$= \frac{(\underline{x}, \underline{y})}{(\underline{y}, \underline{y})}\,\underline{y}$$

Example of orthogonal basis theorem in $\mathbb{R}^2$:

Let $\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$.

Check that $\underline{v}_1$ and $\underline{v}_2$ are $\perp$ and find the coeff α_i in:

$$\sum_{i=1}^{2} \alpha_i \underline{v}_i = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \equiv \underline{x}$$

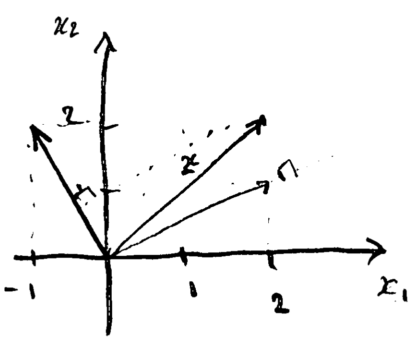
$(\underline{v}_1, \underline{v}_2) = -2 + 2 = 0$ so $\underline{v}_1$ is $\perp$ to $\underline{v}_2$.

$$\alpha_1 = \frac{(\underline{x}, \underline{v}_1)}{(\underline{v}_1, \underline{v}_1)} = \frac{6}{5}$$

$$\alpha_2 = \frac{(\underline{x}, \underline{v}_2)}{(\underline{v}_2, \underline{v}_2)} = \frac{2}{5}$$

Let's check that these coeff. work:

$$\frac{6}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 12 - 2 \\ 6 + 4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



> Any vector in $\mathbb{R}^n$ is the sum of its projections along n $\perp$ vectors.