# MATH 3150 - PDEs for Engineers



## § 1.1 - 1.2, 3.1

Recall partial derivative:

$u(x,t)$ = function of two variables (for example)

$\frac{\partial u(x_0,t_0)}{\partial x}$ = rate of change of $u$ when $x$ changes at $(x,t) = (x_0, t_0)$

= derivative of $u(x, t_0)$ as a function of $x$ evaluated at $x_0$.

$$= \lim_{x \to x_0} \frac{u(x,t_0) - u(x_0,t_0)}{x - x_0}$$

$\frac{\partial u}{\partial t}(x_0, t_0)$ = rate of change of $u$ when $t$ changes at $(x,t) = (x_0, t_0)$.

Many physical laws are described by relations between rates of changes $\rightsquigarrow$ Partial Differential Equations.

This class is to learn how to solve some classic examples of PDEs.

Here are some of them:

## Examples of PDEs

| PDE | model | order | lin |
|---|---|---|---|
| $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ | advection eq. | 1 | y |
| $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ | wave eq. | 2 | y |
| $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ | heat eq. | 2 | y |
| $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ | Burger's eq (1D version of Navier-Stokes i.e. fluid dyn.) | 1 | n |
| $\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{= \Delta u} = 0$ | Laplace eq. diffusion, currents flow in porous medium | 2 | y |
| $\Delta u = f$ | Poisson eq. | 2 | y |

**Order of PDE** $\equiv$ highest order of differentiation

**linear equation**: $L(u) = f$ where

$L$ = linear diff. op. i.e. satisfies, for all $\alpha, \beta \in \mathbb{R}$, $u, v$ functions:

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$$

Examples:

- $L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is linear since

$$L(\alpha u + \beta v) = \frac{\partial^2(\alpha u + \beta v)}{\partial x^2} + \frac{\partial^2(\alpha u + \beta v)}{\partial y^2}$$

$$= \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \beta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$= \alpha L(u) + \beta L(v)$$

- $L(u) = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$ is linear (check)

- $L(u) = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$ is non linear

$$L(\alpha u + \beta v) = \alpha\frac{\partial u}{\partial t} + \beta\frac{\partial v}{\partial t} + (\alpha u + \beta v)\frac{\partial(\alpha u + \beta v)}{\partial x}$$

$$\neq \alpha\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \beta\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right)$$

Homogeneous equation: means all terms in PDE involve $u$.

Example: $L(u) = f$ is homogeneous iff $f = 0$

Poisson eq is linear not homog.

Laplace eq is linear and ~~not~~ homog.

Advection eq:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad \text{(linear homog first order PDE)}$$

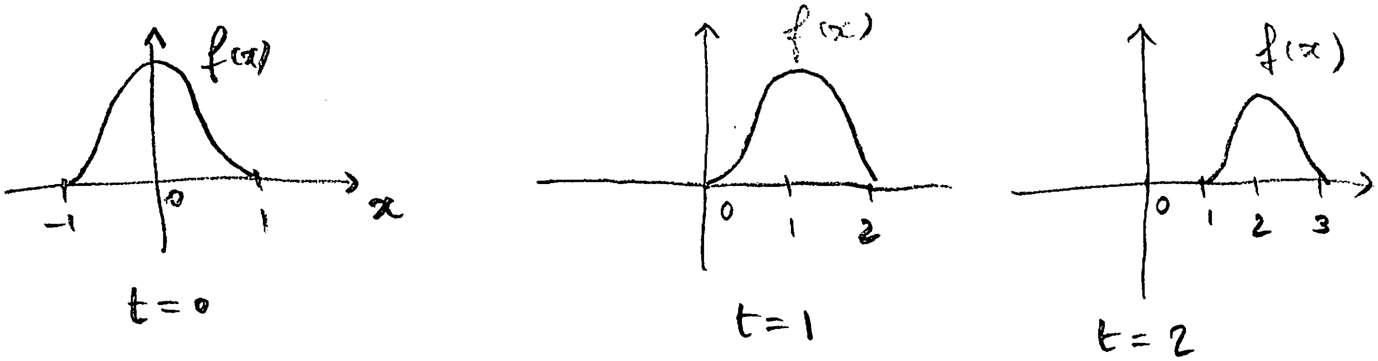
has a sol of the form $\boxed{u(x,t) = f(x-t)}$

Check: $\frac{\partial u}{\partial t} = -f'(x-t)$, $\frac{\partial u}{\partial x} = f'(x-t)$

When working w/ time one usually refers to <u>initial condition</u>:

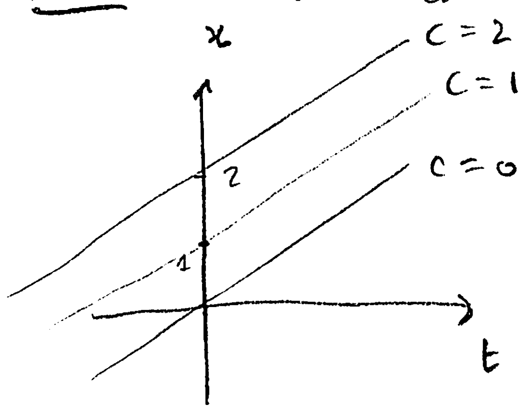
$$u(x,0) = f(x) = \text{initial condition}$$

(how system was at $t=0$)




i.e. origin of I.C. $f(x)$ is changed to $t$
the result is a travelling / propagating wave form

<u>Note</u>: Sol is constant on lines $x - t = c$




these are the <u>characteristic curves</u>
or simply <u>characteristics</u> of PDE

One way of solving PDEs is to find characteristics

Let's apply this method of characteristics to a PDE of the form:

$$(*) \quad \frac{\partial u}{\partial x} + p(x,y)\frac{\partial u}{\partial y} = 0$$

linear and homogeneous first order.

non-constant coeff.

Goal: find characteristics i.e. curves in $xy$ plane where $u(x,y) =$ constant.

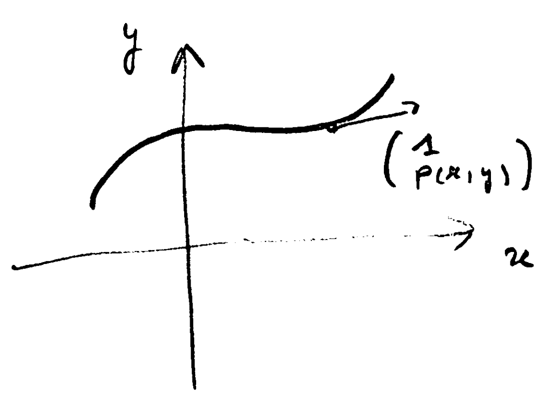
$(*)$ can be rewritten as:

$$\nabla u \cdot \begin{pmatrix} 1 \\ p(x,y) \end{pmatrix} = 0$$

$\Leftrightarrow$ dir. deriv of $u$ in direction $\begin{pmatrix} 1 \\ p(x,y) \end{pmatrix} = 0$

$\Leftrightarrow$ $u(x,y)$ does not change in dir. $\begin{pmatrix} 1 \\ p(x,y) \end{pmatrix}$

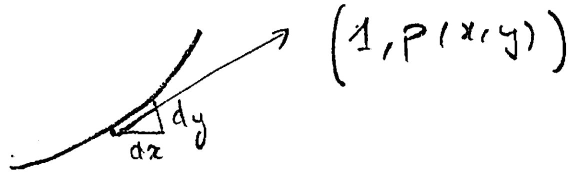
$\Rightarrow$ $\begin{pmatrix} 1 \\ p(x,y) \end{pmatrix}$ = vector tangent to characteristics.



1) <u>Find characteristics:</u>

Solve ODE:

$$\frac{dy}{dx} = \frac{p(x,y)}{1}$$




to get family of curves:

$$y = g(x,y) + c$$

where c is arbitrary integration const.

In general we may write characteristics in parametric form:

$$\Phi(x,y) = y - g(x) = c$$

2) <u>To solve PDE(*)</u>: If $f$ is an arbitrary (need smooth) function:

$$u(x,y) = f(c) = f(\Phi(x,y))$$

some constant

Thus solutions to PDE (*) have form

$$u(x,y) = f(\Phi(x,y))$$

Examples:

- Advection eq: $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ ( $tx$ plane )

characteristics:

$$\frac{dx}{dt} = 1 \Rightarrow x = t + c$$
$$\Rightarrow \Phi(t,x) = x - t = c$$

Solutions: $u(x,t) = f(x-t)$, $f$ arbitrary.

- $\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$

characteristics:

$$\frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2} + c$$
$$\Rightarrow \Phi(x,y) = y - \frac{x^2}{2} = c$$

Solutions: $u(x,y) = f(c) = f\left(y - \frac{x^2}{2}\right)$

check:
$$\frac{\partial u}{\partial x} = -x f'\left(y - \frac{x^2}{2}\right)$$
$$\frac{\partial u}{\partial y} = f'\left(y - \frac{x^2}{2}\right)$$
$$\Rightarrow \frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Characteristics: $(x \neq 0)$

$$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{d(\ln y)}{dx} = \frac{1}{x}$$
$$\Rightarrow \ln y = \ln x + c$$

$$\rightarrow y = x e^c$$ call C again.

$$\Rightarrow \Phi(x,y) = \frac{y}{x} = c$$

Check solutions are $u(x,y) = f\left(\frac{y}{x}\right)$