§ 1.1- 1.2, 3.1

Recall partial derivative:

\( u(x,t) \) = function of two variables (for example)

\[ \frac{\partial u}{\partial x} = \text{rate of change of } u \text{ when } x \text{ changes} \]

at \((x,t) = (x_0, t_0)\)

\[ = \text{derivative of } u(x, t_0) \text{ as a function of } x \]

evaluated at \(x_0\).

\[ = \lim_{x \to x_0} \frac{u(x, t_0) - u(x_0, t_0)}{x - x_0} \]

\( \frac{\partial u}{\partial t} = \text{rate of change of } u \text{ when } t \text{ changes} \)

at \((x,t) = (x_0, t_0)\).

Many physical laws are described by relations between rates of changes in partial Differential Equations.

This class is to learn how to solve some classic examples of PDEs.

Here are some of them:
### Examples of PDEs

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<th>Model</th>
<th>Order</th>
<th>Diagram</th>
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<tr>
<td>( \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 )</td>
<td>advection eq.</td>
<td>1</td>
<td><img src="advection.png" alt="Diagram" /></td>
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<tr>
<td>( \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} )</td>
<td>wave eq.</td>
<td>2</td>
<td><img src="wave.png" alt="Diagram" /></td>
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<td>( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} )</td>
<td>heat eq.</td>
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<td>( \frac{\partial u}{\partial t} + \nu \frac{\partial u}{\partial x} = 0 )</td>
<td>Burgers eq.</td>
<td>1</td>
<td><img src="burgers.png" alt="Diagram" /></td>
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<tr>
<td>(1D version of Navier-Stokes i.e. fluid dyn.)</td>
<td></td>
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<td>( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 )</td>
<td>Laplace eq.</td>
<td>2</td>
<td><img src="laplace.png" alt="Diagram" /></td>
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<tr>
<td>( = \Delta u )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Delta u = f )</td>
<td>Poisson eq.</td>
<td>2</td>
<td><img src="poisson.png" alt="Diagram" /></td>
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</table>

**Order of PDE** = highest order of differentiation

**Linear equations**: \( L(u) = f \) where

\[ L = \text{linear diff. op. i.e. satisfies for all } \alpha, \beta \in \mathbb{R}, \]
\[ u, v \text{ functions: } L(\alpha u + \beta v) = \alpha L(u) + \beta L(v) \]
Example:

\[ L(x) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \]

\[ L(\omega u + \beta v) = \frac{\partial^2 (\omega u + \beta v)}{\partial x^2} + \frac{\partial^2 (\omega u + \beta v)}{\partial y^2} = \omega \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \beta \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = \omega L(u) + \beta L(v) \]

The function \( L(x) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) is linear since

\[ L(\omega u + \beta v) = \omega L(u) + \beta L(v) \]

Homogeneous equation: meaning all terms in PDE involve \( u \).

Poisson eqn is homogeneous iff \( f = 0 \)

Laplace eqn is linear and not Homog.

Check:

\[ \frac{\partial^2 u}{\partial t^2} = -f(x-t), \quad \frac{\partial u}{\partial t} = g(x-t) \]

has a sol of the form

\[ u(x,t) = \phi(x-t) + \psi(x+t) \]

\[ \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial x} \]

[Continued on next page]
When working with time one usually refers to initial condition:

\[ u(x,0) = f(x) = \text{initial condition} \]

(Here system was at \( t = 0 \))

\[ f(x) \]

\[ t = 0 \]

\[ f(x) \]

\[ t = 1 \]

\[ f(x) \]

\[ t = 2 \]

i.e. origin of I.C. \( f(x) \) is changed to \( t \)

the result is a travelling wave front propagating

Note: \( \text{Sol is constant on line} \ x - t = c \)

\[ x \]

\[ c = 2 \]

\[ c = 1 \]

\[ c = 0 \]

these are the characteristic curves

\[ t \]

or simply characteristics of PDE
One way of solving PDEs is to find characteristics of the PDE of the form:

\[ \frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{dz}{c(x,y)} \]

where the characteristic curves are given by:

\[ \frac{dx}{du} = a(x,y), \quad \frac{dy}{du} = b(x,y), \quad \frac{dz}{du} = c(x,y) \]

and can be rewritten as:

\[ \frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{dz}{c(x,y)} \]

where the characteristic curves are given by:

\[ \frac{dx}{du} = a(x,y), \quad \frac{dy}{du} = b(x,y), \quad \frac{dz}{du} = c(x,y) \]

where \( a(x,y) \), \( b(x,y) \), and \( c(x,y) \) are non-constant coefficients.

Goal: find characteristic curves in the plane.

let's apply this method of characteristics to a PDE of the form:

\[ \frac{d}{dx}(a(x,y) \frac{du}{dx} + b(x,y) \frac{du}{dy}) = 0 \]

where \( a(x,y) \), \( b(x,y) \), and \( c(x,y) \) are linear and homogeneous first-order.
1) **Find characteristics:**

Solve ODE:

\[
\frac{dy}{dx} = \frac{p(x,y)}{1}
\]

To get family of curves:

\[
y = g(x, y) + c
\]

where \(c\) is arbitrary integration constant.

In general we may write characteristics in parametric form:

\[
Φ(x, y) = y - g(x) = c
\]

2) **To solve PDE(∗):** If \(f\) is an arbitrary (well smooth) function

\[
\nu(x, y) = f(c) = f(Φ(x, y))
\]

Thus solutions to PDE in (*) have form

\[
\nu(x, y) = f(Φ(x, y))
\]
Examples:

- Advection eq: \( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \) (x,t plane)

  characteristics:
  \[
  \frac{dx}{dt} = 1 \quad \Rightarrow \quad x = t + c \\
  \Rightarrow \quad \Phi(t,x) = x - t = c
  \]

  solutions: \( u(x,t) = f(x-t) \), arbitrary

- \( \frac{\partial^2 u}{\partial x^2} + \frac{x}{y} \frac{\partial u}{\partial y} = 0 \)

  characteristics:
  \[
  \frac{dy}{dx} = x \quad \Rightarrow \quad y = \frac{x^2}{2} + c \\
  \Rightarrow \quad \Phi(x,y) = y - \frac{x^2}{2} = c
  \]

  solutions: \( u(x,y) = f(c) = f(y - \frac{x^2}{2}) \)

  check:
  \[
  \frac{\partial u}{\partial x} = -x \ f'(y - \frac{x^2}{2}) \\
  \frac{\partial u}{\partial y} = \ f'(y - \frac{x^2}{2})
  \]

  \( \Rightarrow \) \( \frac{\partial^2 u}{\partial x^2} + \frac{x}{y} \frac{\partial u}{\partial y} = 0 \)

- \( \frac{x}{y} \frac{\partial u}{\partial x} + \frac{y}{y} \frac{\partial u}{\partial y} = 0 \)

  (x ≠ 0)

  characteristics:
  \( y = xe^c \)

  \[
  \frac{dy}{dx} = \frac{4}{x} \quad \Rightarrow \quad \frac{d(ln y)}{dx} = \frac{1}{x} \\
  \Rightarrow \quad ln y = ln x + c \\
  \Rightarrow \quad \Phi(x,y) = \frac{y}{x} = c
  \]

  check solutions are \( u(x,y) = f(\frac{y}{x}) \)

- call C again.