MATH 3150 S12
PRACTICE FINAL SOLUTIONS

Problem 1 The steady state temperature distribution $u(x)$ must satisfy: $u''=0 \Rightarrow \underline{u(x) = ax+b}$

(a) $u(0)=0 = a\times 0+b$, $u(L)=2 = aL+b$ $\Big\} \Rightarrow$ $b=0$, $a=\frac{2}{L}$

$$\Rightarrow \boxed{u(x) = \frac{2}{L}x}$$

(b) $u'(0)=1 = a$

$u(L)=3 = aL+b \Rightarrow b=3-L$

$$\Rightarrow \boxed{u(x) = x+3-L}$$

(c) $u(0)=2 = a\times 0+b$

$u(L)+u'(L) = 0 = aL+b+a$
$= a(L+1)+2$ $\Big\} \Rightarrow a = \frac{-2}{L+1}$

$$\boxed{u(x) = -\frac{2}{L+1}x+2}$$

Problem 2 $f(r,\theta) = r^4\cos 4\theta$, $\Delta u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}$

$f_{rr} = 12r^2\cos 4\theta$

$f_r = 4r^3\cos 4\theta$

$f_{\theta\theta} = -16r^4\cos 4\theta$

$$f_{rr}+\frac{1}{r}f_r+\frac{1}{r^2}f_{\theta\theta} = 12r^2\cos 4\theta + 4r^2\cos 4\theta - 16r^2\cos 4\theta = 0$$

$\Rightarrow$ f solves Laplace equation.

Problem 3

D'Alembert's solution to:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0, \ t > 0 \\ u(x,0) = \sin\left(\frac{2\pi}{L}x\right), \ x \in [0,L] \\ u_t(x,0) = 0 \end{cases}$$

ds:

$$\boxed{u(x,t) = \frac{1}{2}\left(\sin\left(\frac{2\pi}{L}(x+ct)\right) + \sin\left(\frac{2\pi}{L}(x-ct)\right)\right)}$$

Problem 4

(a) $u(r,\theta) = R(r)\Theta(\theta)$ into Laplace eq:

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$r^2R''\Theta + rR'\Theta + R\Theta'' = 0$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \text{const indep of } r \text{ and } \theta = \lambda$$

$$\Rightarrow r^2R'' + rR' - \lambda R = 0$$

$$\Theta'' + \lambda\Theta = 0$$

(b) $\lambda = n^2 \Rightarrow \boxed{\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad n = 0,1,2,\ldots}$

$R_n(r)$ solves:

$$r^2R_n'' + rR_n' - n^2R_n = 0$$ (Euler eq. solution given in problem formulation)

$$\Rightarrow R_n(r) = \begin{cases} c_1 r^n + c_2 r^{-n}, & n > 0 \\ c_1 + c_2 \ln r, & n = 0 \end{cases}$$

However $R_n(r)$ must be bounded at origin $\Rightarrow c_2 = 0$

(otherwise $|R_n(r)| \to \infty$ as $r \to 0$, since $|r^{-n}| \to \infty$ and $|\ln r| \to \infty$ as $r \to 0$.)

$$\Rightarrow \boxed{R_n(r) = cr^n}, \quad n = 0,1,2,\ldots$$

(c) The functions:

$$u_n(r,\theta) = \begin{cases} r^n(a_n \cos n\theta + b_n \sin n\theta), & n>0 \\ a_0 & , n=0 \end{cases}$$

solve the problem (1) (by separation of variables).

Since problem (1) is linear, a general solution is:

$$\boxed{u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)}$$

To find the coefficients a_n and b_n we simply recognize that:

$$f(\theta) = u(1,\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

= Fourier series of $f(\theta)$ on interval $[0,2\pi]$

$$\Rightarrow \boxed{a_0 = \frac{(1,f)}{(1,1)} = \frac{\int_0^{2\pi} f(\theta)\,d\theta}{2\pi}}$$

$$\boxed{a_n = \frac{(\cos n\theta, f)}{(\cos n\theta, \cos n\theta)} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta, \quad n>0}$$

$$\boxed{b_n = \frac{(\sin n\theta, f)}{(\sin n\theta, \sin n\theta)} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta, \quad n>0}$$

(d) When $f(\theta) = \sin 2\theta$, f is already given as Fourier series.

$$\Rightarrow \quad a_n = 0, \quad n = 0, 1, 2, \ldots$$

$$b_n = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

$$\Rightarrow \boxed{u(r,\theta) = r^2 \sin 2\theta}$$

(e) In cartesian coordinates: $r^2 = x^2 + y^2$

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$= 2 \frac{y}{\sqrt{x^2+y^2}} \frac{x}{\sqrt{x^2+y^2}} = 2 \frac{xy}{x^2+y^2}$$

$$\Rightarrow \boxed{u(x,y) = r^2 \sin 2\theta = 2xy}$$

(check: $\Delta u = u_{xx} + u_{yy} = 0$

$u|_{r=1} = 2 \sin\theta \cos\theta = \sin 2\theta$)

Problem 5

$$\begin{cases} u_t = u_{xx} \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

(a) $u(x,t) = X(x)T(t)$ ansatz into PDE gives:

$$XT' = X''T$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{T'}{T} = \text{constant } -\lambda$$

$$\Rightarrow \quad X'' + \lambda X = 0 \quad , \quad X'(0) = 0, X'(1) = 0$$

and $T' + \lambda T = 0$

Since temperature decays or stays constant we need $\lambda = \mu^2 \geq 0$

$$\Rightarrow X(x) = a \cos \mu x + b \sin \mu x$$

Assuming $\mu \neq 0$:

$$X'(x) = -a\mu \sin \mu x + b\mu \cos \mu x$$

$$X'(0) = 0 = b\mu \underset{\mu>0}{\Rightarrow} \underline{b = 0}$$

$$X'(1) = -a\mu \sin \mu = 0$$

$$\Rightarrow \begin{cases} a = 0 \text{ (trivial sol)} \\ \text{or} \\ \sin \mu = 0 \Rightarrow \mu = \mu_n = n\pi \end{cases}$$

$$\Rightarrow X_n(x) = a_n \cos n\pi x$$

Assuming $\mu = 0$: $X_0(x) = a_0$

$$\Rightarrow \quad T_n(t) = \exp\left[-(n\pi)^2 t\right] \text{ (which works for } n=0)$$

Thus by superposition principle:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \exp\left[-(n\pi)^2 t\right].$$

Now using I. C.:

$$\underset{\substack{\parallel \\ f(x)}}{u(x,0)} = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$\Rightarrow$ a_n are cosine series coeff of $f(x)$.

$$\Rightarrow \quad a_0 = \frac{(f,1)}{(1,1)} = \boxed{\frac{\int_0^1 f(x)\,dx}{1}}$$

$$a_n = \frac{(f, \cos n\pi x)}{(\cos n\pi x, \cos n\pi x)} = \boxed{2\int_0^1 f(x) \cos n\pi x\, dx}$$

(b) With $f(x) = 100x$:

$$a_0 = \int_0^1 (100x)\,dx = 100 \left.\frac{x^2}{2}\right|_0^1 = \boxed{50}$$

$$a_n = 2\int_0^1 (100x) \cos(n\pi x)\, dx$$

$$\overset{\text{IBP}}{=} \underbrace{\left. 200x \frac{\sin n\pi x}{n\pi}\right|_0^1}_{=0} - 200\int_0^1 \frac{\sin n\pi x}{n\pi}\, dx$$

$$= -\frac{200}{n\pi} \left.\frac{(-\cos n\pi x)}{n\pi}\right|_0^1 = \boxed{\frac{200}{(n\pi)^2}\left((-1)^n - 1\right)}$$

(c) $\begin{cases} v_t = v_{xx} \\ v_x(0,t) = v_x(1,t) = 1 \\ v(x,0) = g(x) \end{cases}$

Let $v(x,t) = u(x,t) + x$.

$\Rightarrow \left.\begin{array}{l} v_t = u_t \\ v_{xx} = u_{xx} \end{array}\right\} \Rightarrow \boxed{v_t = v_{xx}}$

$\boxed{\begin{array}{l} v_x(0,t) = u_x(0,t) + 1 = 1 \\ v_x(1,t) = u_x(1,t) + 1 = 1 \end{array}}$

$\boxed{v(x,0) = u(x,0) + x = \underbrace{100x}_{f(x)} + x = 101x}$

$\Rightarrow v(x,t)$ solves (5)

Problem 6

(a) $X'' + \mu^2 X = 0$, $X(0) = 0$, $X(1) = 0$

$\Rightarrow X(x) = a\cos\mu x + b\sin\mu x.$

$X(0) = a = 0$

$X(1) = b\sin\mu = 0 \Rightarrow \begin{cases} b = 0 & \text{(trivial sol)} \\ \mu_m = m\pi, & m = 1, 2, \ldots \end{cases}$

$\Rightarrow \boxed{X_m(x) = \sin m\pi x}$, $m = 1, 2, \ldots$

Similarly: $\nu_n = n\pi$, $n = 1, 2, \ldots$

$\boxed{Y_n(y) = \sin n\pi y}$, $n = 1, 2, 3, \ldots$

Hence the equation for T becomes:

$$\begin{cases} T''_{m,n} + \overbrace{\left((m\pi)^2 + (n\pi)^2\right)}^{\lambda_{mn}^2} T = 0 \\ T'_{m,n}(0) = 0 \end{cases}$$

$$\Rightarrow \quad T_{mn}(t) = a_{mn} \cos \lambda_{mn} t + b_{mn} \sin \lambda_{mn} t$$

$$T'_{mn}(t) = -\lambda_{mn} a_{mn} \sin \lambda_{mn} t + b_{mn} \lambda_{mn} \cos \lambda_{mn} t$$

$$T'_{m}(0) = b_{mn} \lambda_{mn} = 0 \underset{\lambda_{mn} \neq 0}{\Rightarrow} \underline{b_{mn} = 0}$$

$$\Rightarrow \boxed{T_{mn} = a_{mn} \cos \lambda_{mn} t.}$$

Product solutions are:

$$\boxed{u_{m,n}(x,y,t) = B_{mn} \cos(\lambda_{mn} t) \sin(m\pi x) \sin(n\pi y)}$$

ok we just renamed this const.

(b) General form is by superposition principle:

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \cos(\lambda_{mn} t) \sin(m\pi x) \sin(n\pi y)$$

$$\underset{f(x,y)}{\underset{\parallel}{u(x,y,0)}} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin(m\pi x) \sin(n\pi y) \text{ - double sine series}$$

$$\Rightarrow \boxed{B_{mn}} = \frac{\left(f(x,y),\ \sin(m\pi x)\sin(n\pi y)\right)}{\left(\sin(m\pi x)\sin(n\pi y),\ \sin(m\pi x)\sin(n\pi y)\right)}$$

$$= \boxed{4\int_0^1\int_0^1 f(x,y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy}$$

using §0.1

(c) Let $f(x,y) = x(1-x)\,y(1-y)$

$$\Rightarrow \boxed{B_{mn}} = 4\int_0^1 dx\int_0^1 dy\ x(1-x)\sin(m\pi x)\,y(1-y)\sin(n\pi y)$$

$$= 4\left(\int_0^1 dx\ x(1-x)\sin(m\pi x)\right)\left(\int_0^1 dy\ y(1-y)\sin(n\pi y)\right)$$

$$= 4\ \frac{2\left((-1)^m-1\right)}{\pi^3 m^3}\ \frac{2\left((-1)^n-1\right)}{\pi^3 n^3}$$

$$= \boxed{\frac{16\left((-1)^m-1\right)\left((-1)^n-1\right)}{\pi^6 m^3 n^3}}$$

(d) Solution is:

$$\boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{16\left((-1)^m-1\right)\left((-1)^n-1\right)}{\pi^6 m^3 n^3}\sin(m\pi x)\sin(n\pi y)\cdot\cos\left(\sqrt{(m\pi)^2+(n\pi)^2}\,t\right)}$$

Problem 7

$$u(r,\theta,t)=\sum_{n=1}^{\infty} a_{2n}\, J_2(\alpha_{2n} r)\cos 2\theta \exp[-\alpha_{2n}^2 t]$$

(a) For $f_1(r,\theta)$:

$$a_{2n} = \frac{2}{J_3^2(\alpha_{2n})}\left[\int_0^1 J_2(\alpha_{2n} r)\, J_2(\alpha_{21} r)\, r\,dr\right]\underbrace{\left[\frac{1}{\pi}\int_0^{2\pi}\cos^2 2\theta\, d\theta\right]}_{=1}$$

$$= \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

Another way of justifying that a_{2n} is 1 is to recognize:

$$u(r,\theta,t)= J_2(\alpha_{21} r)\cos 2\theta \exp[-\alpha_{21}^2 t]$$

and $u(r,\theta,0)= f_1(r,\theta)$

For $f_2(r,\theta)$ we similarly get:

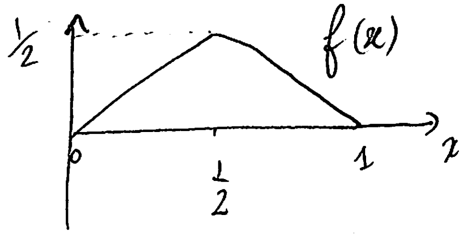
$$u(r,\theta,t)= J_2(\alpha_{22} r)\cos(2\theta)\exp[-\alpha_{22}^2 t]$$

(b) $\alpha_{21} < \alpha_{22} \Rightarrow u_2 \to 0$ faster than u_1.

Problem 8

$$f(x) = \begin{cases} x & \text{if } 0 < x < \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

(a)



(b) Sine series: $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

$$b_n = 2\int_0^1 f(x) \sin n\pi x \, dx$$

$$= 2\int_0^{\frac{1}{2}} x \sin n\pi x \, dx + 2\int_{\frac{1}{2}}^1 (1-x) \sin n\pi x \, dx$$

$$\overset{\text{IBP}}{=} 2 x \left(\frac{-\cos n\pi x}{n\pi}\right)\Big|_0^{\frac{1}{2}} + 2\int_0^{\frac{1}{2}} \frac{\cos n\pi x}{n\pi} dx$$

these terms cancel each other

$$+ 2(1-x)\left(\frac{-\cos n\pi x}{n\pi}\right)\Big|_{\frac{1}{2}}^1 - 2\int_{\frac{1}{2}}^1 \frac{\cos n\pi x}{n\pi} dx$$

$$= \boxed{\frac{4}{(n\pi)^2} \sin \frac{n\pi}{2}}$$

(c) Cosine series: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$

$$a_0 = \int_0^1 f(x)\,dx$$

$$a_n = 2\int_0 f(x) \cos(n\pi x)\,dx$$

$$a_0 = \int_0^1 f(x)\,dx = \int_0^{\frac{1}{2}} x\,dx + \int_{\frac{1}{2}}^1 (1-x)\,dx$$

$$= \frac{x^2}{2}\Big|_0^{\frac{1}{2}} + x - \frac{x^2}{2}\Big|_{1/2}^1$$

$$= \frac{1}{8} + \frac{1}{8} = \boxed{\frac{1}{4}}$$

$$a_n = 2\int_0^{\frac{1}{2}} x \cos n\pi x\,dx + 2\int_{\frac{1}{2}}^1 (1-x)\cos n\pi x\,dx$$

$$\overset{IBP}{=} 2x\,\frac{\sin n\pi x}{n\pi}\Big|_0^{\frac{1}{2}} - 2\int_0^{\frac{1}{2}} \frac{\sin n\pi x}{n\pi}\,dx$$

cancel each other

$$+ 2(1-x)\frac{\sin n\pi x}{n\pi}\Big|_{\frac{1}{2}}^1 + 2\int_{\frac{1}{2}}^1 \frac{\sin n\pi x}{n\pi}\,dx$$

$$= \frac{2\cos n\pi x}{(n\pi)^2}\Big|_0^{\frac{1}{2}} - 2\frac{\cos n\pi x}{(n\pi)^2}\Big|_{\frac{1}{2}}^1$$

$$= \boxed{\left(4\cos\frac{n\pi}{2} - 2 - 2(-1)^n\right)\frac{1}{(n\pi)^2}}$$

(d) Fourier series: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x$

with a_n and b_n as above.