Math 3150-4, Practice Final
Spring 2012

Total points: 130/120.
Notes: Problems are independent of each other. This practice exam is longer and more difficult than the actual exam.

Problem 1 (10 pts) Consider a bar of length $L$. The position on the bar is given by $x \in [0, L]$. Find the steady state temperature distribution $u(x)$ in the following situations:
(a) $u(0) = 0$ and $u(L) = 2$.
(b) $u'(0) = 1$ and $u(L) = 3$.
(c) $u(0) = 2$ and $u(L) + u'(L) = 0$.

Problem 2 (10 pts) Check whether $f(r, \theta) = r^4 \cos(4\theta)$ (in polar coordinates) satisfies the Laplace equation $\Delta u = 0$.

Problem 3 (10 pts) Consider a string of length $L$ with position $x \in [0, L]$. Give d’Alembert’s solution to the wave equation
\[
\begin{cases}
  u_{tt} = c^2 u_{xx}, & x \in [0, L], \ t > 0 \\
  u(0, t) = u(L, t) = 0, & t > 0 \\
  u(x, 0) = \sin(\frac{2\pi}{L}x), & x \in [0, L] \\
  u_t(x, 0) = 0, & x \in [0, L].
\end{cases}
\]

Problem 4 (20 pts) Consider the Dirichlet problem on the unit disk,
\[
\begin{cases}
  \Delta u = 0, & 0 < r < 1, 0 < \theta < 2\pi \\
  u(1, \theta) = f(\theta), & 0 < \theta < 2\pi.
\end{cases}
\]
Recall that
- The Laplacian in polar coordinates is $\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$.
- A general form of the solution to the ODE $x^2 y'' + xy' + \rho^2 y = 0$ is
  \[
  y(x) = c_1 x^\rho + c_2 x^{-\rho}, \quad \text{if } \rho \neq 0,
  \]
  \[
  y(x) = c_1 x + c_2 \ln x, \quad \text{if } \rho = 0.
  \]
(a) Use separation of variables with $u(r, \theta) = R(r)\Theta(\theta)$ to show that the separated equations are of the form
\[
\begin{align*}
  r^2 R'' + r R' - \lambda R &= 0, \\
  \Theta'' + \lambda \Theta &= 0.
\end{align*}
\]
(b) Since $\Theta$ needs to be $2\pi$-periodic, $\lambda = n^2$, $n = 0, 1, 2, \ldots$.
Solve equations (2) and (3).
(c) Show that the general form of the solution to the Dirichlet problem (1) is
\[
u(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta).
\]
Specify what are the coefficients $a_n$ and $b_n$ in terms of $f(\theta)$.
(d) Solve the Dirichlet problem (1) with $f(\theta) = \sin(2\theta)$.
(e) [Extra credit] Write the solution to (d) in Cartesian coordinates.
**Problem 5 (20 pts)** Consider the 1D heat equation with homogeneous Neumann boundary conditions modeling a bar with insulated ends:

\[
\begin{cases}
  u_t = u_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0, \\
  u_x(0, t) = u_x(1, t) = 0 & \text{for } t > 0, \\
  u(x, 0) = f(x), & \text{for } 0 < x < 1.
\end{cases}
\]

(a) Use separation of variables with \( u(x, t) = X(x)T(t) \) to show that a general solution to (4) is

\[ u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \exp[-(n\pi)^2 t]. \]

Specify what the coefficients \( a_n, n = 0, 1, 2, \ldots \) are in terms of \( f(x) \).

(b) Solve (4) with \( f(x) = 100x \).

(c) Now consider the following 1D heat equation with inhomogeneous Neumann boundary conditions:

\[
\begin{cases}
  v_t = v_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0, \\
  v_x(0, t) = v_x(1, t) = 1 & \text{for } t > 0, \\
  v(x, 0) = g(x), & \text{for } 0 < x < 1
\end{cases}
\]

Show that \( v(x, t) = u(x, t) + x \) solves (5) with \( g(x) = f(x) + x \) and \( u(x, t) \) as in (b).

**Problem 6 (20 pts)** Consider the 2D wave equation below which models the vibrations of square membrane with fixed edges, initial position \( f(x, y) \) and zero initial velocity.

\[
\begin{cases}
  u_{tt} = u_{xx} + u_{yy}, & \text{for } 0 < x < 1, 0 < y < 1 \text{ and } t > 0, \\
  u(0, y, t) = u(1, y, t) = 0, & \text{for } 0 < y < 1 \text{ and } t > 0 \\
  u(x, 0, t) = u(x, 1, t) = 0, & \text{for } 0 < x < 1 \text{ and } t > 0 \\
  u(x, y, 0) = f(x, y), & \text{for } 0 < x < 1, 0 < y < 1 \\
  u_t(x, y, 0) = 0, & \text{for } 0 < x < 1, 0 < y < 1.
\end{cases}
\]

Separation of variables with \( u(x, y, t) = X(x)Y(y)T(t) \) gives the ODEs:

\[ X'' + \mu^2 X = 0, \quad X(0) = 0, \quad X(1) = 0 \]
\[ Y'' + \nu^2 Y = 0, \quad Y(0) = 0, \quad Y(1) = 0 \]
\[ T'' + (\mu^2 + \nu^2) T = 0, \quad T'(0) = 0. \]

(a) Obtain the product solutions

\[ u_{m,n}(x, y, t) = B_{m,n} \cos(\lambda_{m,n} t) \sin(m\pi x) \sin(n\pi y). \]

where \( \lambda_{m,n} = \sqrt{(m\pi)^2 + (n\pi)^2} \). **Note:** The ODE’s for \( X \) and \( Y \) are very similar. Solving one of them in detail and stating the result for the other one should be enough.

(b) Write down the general form of a solution \( u(x, y, t) \) to (6). Use initial conditions and orthogonality of double sine series to express \( B_{m,n} \) in terms of \( f(x, y) \).

(c) Using that

\[ \int_0^1 x(1-x) \sin(m\pi x)dx = \frac{2((-1)^m - 1)}{\pi^3 m^3}, \]
find the coefficients $B_{m,n}$ in the double sine series,

$$x(1-x)y(1-y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{m,n} \sin(m\pi x) \sin(n\pi y).$$

(d) Solve 2D wave equation (6) with $f(x, y) = x(1-x)y(1-y)$.

**Problem 7 (20 pts)** Consider a circular plate of radius 1 with initial temperature distribution of the form $f(r, \theta) = g(r) \cos 2\theta$ and where the outer rim of the plate is kept in an ice bath. The temperature distribution $u(r, \theta, t)$ satisfies the 2D Heat equation

$$u_t = \Delta u \quad \text{for } 0 < r < 1, \ 0 \leq \theta \leq 2\pi \text{ and } t > 0$$

$$\begin{cases} 
  u(r, \theta, 0) = f(r, \theta) & \text{for } 0 < r < 1 \text{ and } 0 \leq \theta \leq 2\pi \\
  u(1, \theta, t) = 0 & \text{for } 0 \leq \theta \leq 2\pi \text{ and } t > 0
\end{cases}$$

Because the initial temperature distribution is a multiple of $\cos 2\theta$, the solution can be shown to be

$$u(r, \theta, t) = \sum_{n=1}^{\infty} a_{2n} J_2(\alpha_{2n} r) \cos 2\theta \exp[-\alpha_{2n}^2 t].$$

where $\alpha_{2n}$ denotes the $n-$th zero of the Bessel function of the first kind of order 2, and

$$a_{2n} = \frac{2}{\pi J_{2+1}(\alpha_{2n})} \int_0^1 \int_0^{2\pi} f(r, \theta) J_2(\alpha_{2n} r) \cos 2\theta \ d\theta \ dr \quad \text{for } n = 1, 2, \ldots$$

(a) Solve (7) with the initial temperatures

$$f_1(r, \theta) = J_2(\alpha_{2,1} r) \cos 2\theta \quad \text{and} \quad f_2(r, \theta) = J_2(\alpha_{2,2} r) \cos 2\theta.$$

(b) The steady state temperature distribution is $u = 0$. Of the initial temperatures $f_1(r, \theta)$ and $f_2(r, \theta)$, which decays faster to the steady state? Justify your answer.

**Problem 8 (20 pts)** Consider the function

$$f(x) = \begin{cases} 
  x & \text{if } 0 < x < \frac{1}{2} \\
  1-x & \text{if } \frac{1}{2} < x < 1
\end{cases}$$

(a) Plot the function on the interval $[0, 1]$.

(b) Calculate the sine series of $f(x)$.

(c) Calculate the cosine series of $f(x)$.

(d) Calculate the Fourier series of $f(x)$. 

3
Some useful formulas

0.1. Orthogonality relations for double sine series. With the inner product

\[ (u, v) = \int_0^a \int_0^b u(x, y)v(x, y)dxdy, \]

we have for all \( m, n, m' \) and \( n' \) non-zero integers,

\[
\left( \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right), \sin \left( \frac{m'\pi}{a} x \right) \sin \left( \frac{n'\pi}{b} y \right) \right) = \begin{cases} \frac{ab}{4} & \text{if } m = m' \text{ and } n = n' \\ 0 & \text{if } m \neq m' \text{ or } n \neq n' \end{cases}
\]

0.2. Orthogonality relations for sine series. With the inner product

\[ (u, v) = \int_0^a u(x)v(x)dx, \]

we have for all \( m, n \) non-zero integers,

\[
\left( \sin \left( \frac{m\pi}{a} x \right), \sin \left( \frac{n\pi}{a} x \right) \right) = \begin{cases} \frac{a}{2} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}
\]

0.3. Fourier series. For a \( 2p \)-periodic piecewise smooth function \( f \),

\[ f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \omega_n x + b_n \sin \omega_n x, \]

where \( \omega_n = n\pi/p \) and

\[ a_0 = \frac{(f, 1)}{(1, 1)}, \quad a_n = \frac{(f, \cos \omega_n x)}{(\cos \omega_n x, \cos \omega_n x)}, \quad \text{and} \quad b_n = \frac{(f, \sin \omega_n x)}{(\sin \omega_n x, \sin \omega_n x)}. \]

The inner product is \((u, v) = \int_{-p}^p u(x)v(x)dx\). The orthogonality relations are

\[
(\cos \omega_n x, \cos \omega_m x) = \begin{cases} 2p & \text{if } n = m = 0 \\ p & \text{if } n = m > 0 \\ 0 & \text{if } n \neq m \end{cases},
\]

\[
(\cos \omega_n x, \sin \omega_m x) = 0,
\]

\[
(\sin \omega_n x, \sin \omega_m x) = \begin{cases} p & \text{if } n = m > 0 \\ 0 & \text{if } n \neq m \end{cases}.
\]

0.4. Bessel functions. The following identities are valid for \( p \geq 0 \) and \( n = 0, 1, \ldots \)

\[ \int J_1(r)dr = -J_0(r) + C \quad \text{and} \quad \int r^{p+1}J_p(r)dr = r^{p+1}J_{p+1}(r) + C \]

For \( k \geq 0, a > 0 \) and \( \alpha > 0 \), we have

\[ \int_0^a \left( a^2 - r^2 \right) r^{k+1}J_k \left( \frac{\alpha}{a} r \right) dr = \frac{a^{k+4}}{\alpha^2} J_{k+2}(\alpha). \]
0.5. **Orthogonality relations for Bessel functions.** Let $a > 0$ and $m \geq 0$ be fixed. Denote with $\alpha_{mn}$ the $n$–th positive zero of the Bessel function of the first kind of order $m$. With the inner product

$$(u, v) = \int_0^a u(r)v(r)r \, dr$$

we have for all $j, k$ non-zero integers,

$$(J_m(\frac{\alpha_{mj}}{a} r), J_m(\frac{\alpha_{mk}}{a} r)) = \begin{cases} \frac{a^2}{2}J_{m+1}(\alpha_{mj}) & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$