Math 3150-4, Practice Final
Spring 2012

Total points: 130/120.
Notes: Problems are independent of each other. This practice exam is longer and more difficult than the actual exam.

Problem 1 (10 pts) Consider a bar of length L. The position on the bar is given by $x \in [0, L]$. Find the steady state temperature distribution $u(x)$ in the following situations:
(a) $u(0) = 0$ and $u(L) = 2$.
(b) $u'(0) = 1$ and $u(L) = 3$.
(c) $u(0) = 2$ and $u(L) + u'(L) = 0$.

Problem 2 (10 pts) Check whether $f(r, \theta) = r^4 \cos(4\theta)$ (in polar coordinates) satisfies the Laplace equation $\Delta u = 0$.

Problem 3 (10 pts) Consider a string of length L with position $x \in [0, L]$. Give d'Alembert's solution to the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}, \ x \in [0, L], \ t > 0 \\ u(0, t) = u(L, t) = 0, \ t > 0 \\ u(x, 0) = \sin(\frac{2\pi}{L}x), \ x \in [0, L] \\ u_t(x, 0) = 0, \ x \in [0, L]. \end{cases}$$

Problem 4 (20 pts) Consider the Dirichlet problem on the unit disk,

$$\begin{cases} \Delta u = 0, \ \ 0 < r < 1, \ \ 0 < \theta < 2\pi \\ u(1, \theta) = f(\theta), \ \ 0 < \theta < 2\pi. \end{cases} \tag{1}$$

Recall that
- The Laplacian in polar coordinates is $\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$.
- A general form of the solution to the ODE $x^2 y'' + x y' + \rho^2 y = 0$ is

$$y(x) = c_1 x^{\rho} + c_2 x^{-\rho}, \quad \text{if } \rho \neq 0,$$
$$y(x) = c_1 x + c_2 \ln x, \quad \text{if } \rho = 0.$$

(a) Use separation of variables with $u(r, \theta) = R(r)\Theta(\theta)$ to show that the separated equations are of the form

$$r^2 R'' + r R' - \lambda R = 0, \tag{2}$$
$$\Theta'' + \lambda \Theta = 0. \tag{3}$$

(b) Since Θ needs to be $2\pi-$periodic, $\lambda = n^2$, $n = 0, 1, 2, \ldots$.
Solve equations (2) and (3).
(c) Show that the general form of the solution to the Dirichlet problem (1) is

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta).$$

Specify what are the coefficients a_n and b_n in terms of $f(\theta)$.
(d) Solve the Dirichlet problem (1) with $f(\theta) = \sin(2\theta)$.
(e) [Extra credit] Write the solution to (d) in Cartesian coordinates.

Problem 5 (20 pts) Consider the 1D heat equation with homogeneous Neumann boundary conditions modeling a bar with insulated ends:

$$\begin{cases} u_t = u_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0, \\ u_x(0,t) = u_x(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = f(x), & \text{for } 0 < x < 1. \end{cases} \tag{4}$$

(a) Use separation of variables with $u(x,t) = X(x)T(t)$ to show that a general solution to (4) is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \exp[-(n\pi)^2 t].$$

Specify what the coefficients a_n, $n = 0, 1, 2, \ldots$ are in terms of $f(x)$.

(b) Solve (4) with $f(x) = 100x$.

(c) Now consider the following 1D heat equation with *inhomogeneous* Neumann boundary conditions:

$$\begin{cases} v_t = v_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0, \\ v_x(0,t) = v_x(1,t) = 1 & \text{for } t > 0, \\ v(x,0) = g(x), & \text{for } 0 < x < 1 \end{cases} \tag{5}$$

Show that $v(x,t) = u(x,t) + x$ solves (5) with $g(x) = f(x) + x$ and $u(x,t)$ as in (b).

Problem 6 (20 pts) Consider the 2D wave equation below which models the vibrations of square membrane with fixed edges, initial position $f(x,y)$ and zero initial velocity.

$$\begin{cases} u_{tt} = u_{xx} + u_{yy}, & \text{for } 0 < x < 1,\ 0 < y < 1 \text{ and } t > 0, \\ u(0,y,t) = u(1,y,t) = 0, & \text{for } 0 < y < 1 \text{ and } t > 0 \\ u(x,0,t) = u(x,1,t) = 0, & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u(x,y,0) = f(x,y), & \text{for } 0 < x < 1,\ 0 < y < 1 \\ u_t(x,y,0) = 0, & \text{for } 0 < x < 1,\ 0 < y < 1. \end{cases} \tag{6}$$

Separation of variables with $u(x,y,t) = X(x)Y(y)T(t)$ gives the ODEs:

$$X'' + \mu^2 X = 0,\ X(0) = 0,\ X(1) = 0$$
$$Y'' + \nu^2 Y = 0,\ Y(0) = 0,\ Y(1) = 0$$
$$T'' + (\mu^2 + \nu^2)T = 0,\ T'(0) = 0.$$

(a) Obtain the product solutions

$$u_{m,n}(x,y,t) = B_{m,n} \cos(\lambda_{m,n} t) \sin(m\pi x) \sin(n\pi y).$$

where $\lambda_{m,n} = \sqrt{(m\pi)^2 + (n\pi)^2}$. **Note:** The ODE's for X and Y are very similar. Solving one of them in detail and stating the result for the other one should be enough.

(b) Write down the general form of a solution $u(x,y,t)$ to (6). Use initial conditions and orthogonality of double sine series to express $B_{m,n}$ in terms of $f(x,y)$.

(c) Using that

$$\int_0^1 x(1-x) \sin(m\pi x)\,dx = \frac{2((-1)^m - 1)}{\pi^3 m^3},$$

find the coefficients $B_{m,n}$ in the double sine series,

$$x(1-x)y(1-y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{m,n} \sin(m\pi x)\sin(n\pi y).$$

(d) Solve 2D wave equation (6) with $f(x, y) = x(1-x)y(1-y)$.

Problem 7 (20 pts) Consider a circular plate of radius 1 with initial temperature distribution of the form $f(r,\theta) = g(r)\cos 2\theta$ and where the outer rim of the plate is kept in an ice bath. The temperature distribution $u(r, \theta, t)$ satisfies the 2D Heat equation

$$(7) \qquad \begin{cases} u_t = \Delta u & \text{for } 0 < r < 1,\ 0 \le \theta \le 2\pi \text{ and } t > 0 \\ u(r,\theta,0) = f(r,\theta) & \text{for } 0 < r < 1 \text{ and } 0 \le \theta \le 2\pi \\ u(1,\theta,t) = 0 & \text{for } 0 \le \theta \le 2\pi \text{ and } t > 0 \end{cases}$$

Because the initial temperature distribution is a multiple of $\cos 2\theta$, the solution can be shown to be

$$u(r,\theta,t) = \sum_{n=1}^{\infty} a_{2n} J_2(\alpha_{2n} r)\cos 2\theta \exp[-\alpha_{2n}^2 t].$$

where α_{2n} denotes the $n-$th zero of the Bessel function of the first kind of order 2, and

$$a_{2n} = \frac{2}{\pi J_{2+1}^2(\alpha_{2n})}\int_0^1\int_0^{2\pi} f(r,\theta)J_2(\alpha_{2n}r)\cos 2\theta\, d\theta\, r dr \quad \text{for } n = 1, 2, \ldots$$

(a) Solve (7) with the initial temperatures

$$f_1(r,\theta) = J_2(\alpha_{2,1}r)\cos 2\theta \quad \text{and} \quad f_2(r,\theta) = J_2(\alpha_{2,2}r)\cos 2\theta.$$

(b) The steady state temperature distribution is $u = 0$. Of the initial temperatures $f_1(r,\theta)$ and $f_2(r,\theta)$, which decays faster to the steady state? Justify your answer.

Problem 8 (20 pts) Consider the function

$$f(x) = \begin{cases} x & \text{if } 0 < x < \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

(a) Plot the function on the interval $[0, 1]$.
(b) Calculate the sine series of $f(x)$.
(c) Calculate the cosine series of $f(x)$.
(d) Calculate the Fourier series of $f(x)$.

Some useful formulas

0.1. Orthogonality relations for double sine series. With the inner product

$$(u, v) = \int_0^a \int_0^b u(x, y)v(x, y)dxdy,$$

we have for all m, n, m' and n' non-zero integers,

$$\left(\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), \sin\left(\frac{m'\pi}{a}x\right)\sin\left(\frac{n'\pi}{b}y\right)\right) = \begin{cases} \frac{ab}{4} & \text{if } m = m' \text{ and } n = n' \\ 0 & \text{if } m \neq m' \text{ or } n \neq n' \end{cases}$$

0.2. Orthogonality relations for sine series. With the inner product

$$(u, v) = \int_0^a u(x)v(x)dx,$$

we have for all m, n non-zero integers,

$$\left(\sin\left(\frac{m\pi}{a}x\right), \sin\left(\frac{n\pi}{a}x\right)\right) = \begin{cases} \frac{a}{2} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

0.3. Fourier series. For a $2p-$periodic piecewise smooth function f,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\omega_n x + b_n \sin\omega_n x,$$

where $\omega_n = n\pi/p$ and

$$a_0 = \frac{(f, 1)}{(1, 1)}, \quad a_n = \frac{(f, \cos\omega_n x)}{(\cos\omega_n x, \cos\omega_n x)}, \text{ and } b_n = \frac{(f, \sin\omega_n x)}{(\sin\omega_n x, \sin\omega_n x)}.$$

The inner product is $(u, v) = \int_{-p}^{p} u(x)v(x)dx$. The orthogonality relations are

$$(\cos\omega_n x, \cos\omega_m x) = \begin{cases} 2p & \text{if } n = m = 0 \\ p & \text{if } n = m > 0 \\ 0 & \text{if } n \neq m \end{cases},$$

$$(\cos\omega_n x, \sin\omega_m x) = 0,$$

$$(\sin\omega_n x, \sin\omega_m x) = \begin{cases} p & \text{if } n = m > 0 \\ 0 & \text{if } n \neq m \end{cases}.$$

0.4. Bessel functions. The following identities are valid for $p \geq 0$ and $n = 0, 1, \ldots$.

$$\int J_1(r)dr = -J_0(r) + C \quad \text{and} \quad \int r^{p+1}J_p(r)dr = r^{p+1}J_{p+1}(r) + C$$

For $k \geq 0$, $a > 0$ and $\alpha > 0$, we have

$$\int_0^a (a^2 - r^2)r^{k+1}J_k\left(\frac{\alpha}{a}r\right)dr = 2\frac{a^{k+4}}{\alpha^2}J_{k+2}(\alpha).$$

0.5. **Orthogonality relations for Bessel functions.** Let $a > 0$ and $m \geq 0$ be fixed. Denote with α_{mn} the $n-$th positive zero of the Bessel function of the first kind of order m. With the inner product

$$(u, v) = \int_0^a u(r)v(r)r\, dr$$

we have for all j, k non-zero integers,

$$\left(J_m(\frac{\alpha_{mj}}{a}r), J_m(\frac{\alpha_{mk}}{a}r)\right) = \begin{cases} \frac{a^2}{2} J_{m+1}^2(\alpha_{mj}) & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$