Math 3150-4, Midterm Exam 2
Spring 2012

Name: SOLUTIONS **uNID:** ____________

Total points: 110/100 (subject to change). **Total problems:** 5.
Note: Problems are independent of each other.
Problem 1 (20 pts) Consider the heat equation on a bar of length π with **inhomogeneous Dirichlet boundary conditions**:

$$\text{(HEQ)} \quad \begin{cases} u_t = u_{xx}, & x \in (0,\pi), t > 0 \\ u(0,t) = 0, & t > 0 \\ u(\pi,t) = 100, & t > 0 \\ u(x,0) = 100, & x \in [0,\pi]. \end{cases}$$

The boundary conditions correspond to keeping the left end point in an ice bath and the right end point in a bath of temperature 100^o.

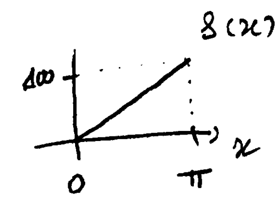
(a) Find the steady state solution $s(x)$ for (HEQ).

(b) Formulate a heat equation with homogeneous Dirichlet boundary conditions that you can use to solve (HEQ). Be sure to specify the **initial condition** and the **relation between the solutions** of the new and old problems.

(c) Solve (HEQ). **Hint:**

$$\int_0^\pi (1 - \frac{x}{\pi}) \sin nx \, dx = \frac{1}{n}.$$

(a) $s_{xx} = 0$ and $s(0) = 0$, $s(\pi) = 100$ $\Longrightarrow$ $s(x) = \frac{100}{\pi} x$



(b) $\begin{cases} v_t = v_{xx} \\ v(0,t) = 0 \\ v(\pi,t) = 0 \\ v(x,0) = 100 - s(x) \end{cases}$ $\quad \underline{v = u - s}$

(c) A sol. to the problem in (b) is:

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) \exp[-n^2 t] \quad \Rightarrow \boxed{u(x,t) = v(x,t) + s(x).}$$

with $b_n = \frac{2}{\pi} \int_0^\pi 100(1 - \frac{x}{\pi}) \sin nx \, dx$

$$= \frac{200}{\pi} \int_0^\pi (1 - \frac{x}{\pi}) \sin nx \, dx$$

$$= \frac{200}{n\pi}$$

Problem 2 (30 pts) The goal of this problem is to solve the Heat Equation with *mixed boundary conditions*

$$\text{(MIX)} \qquad \begin{cases} u_t = u_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u(0,t) = 0 & \text{for } t > 0 \\ u_x(1,t) = 0 & \text{for } t > 0 \\ u(x,0) = f(x) & \text{for } 0 < x < 1 \end{cases}$$

(a) Let $u(x,t) = X(x)T(t)$. Use separation of variables to obtain the separated equations

$X'' + kX = 0$, **(You need to specify the boundary conditions)**

$T' + kT = 0.$

(b) Assuming $k = \lambda^2 > 0$, explain why the only solutions to the ODE of X are of the form $X_n(x) = \sin(\lambda_n x)$, with $\lambda_n = \frac{2n+1}{2}\pi$, for $n = 0, 1, 2, \ldots$.

(c) Show that a general solution to (MIX) is

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin(\lambda_n x) \exp[-\lambda_n^2 t].$$

(d) Consider the inner product $(u,v) = \int_0^1 u(x)v(x)dx$. Given the orthogonality relations valid for $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$

$$(\sin(\lambda_n x), \sin(\lambda_m x)) = \begin{cases} \frac{1}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m, \end{cases}$$

show that

$$b_n = 2\int_0^1 \sin(\lambda_n x) f(x)dx, \quad \text{for } n = 0, 1, 2, \ldots$$

(e) Solve problem (MIX) with $f(x) = \sin(\pi x/2)$.

(a) $u(x,t) = X(x)T(t)$. Plugging into (MIX) we get:

$$XT' = X''T$$

$$\Leftrightarrow \quad \frac{X''}{X} = \frac{T'}{T} = \text{constant } -k$$

$$\Rightarrow \begin{cases} X'' + kX = 0 \\ X(0) = 0,\ X'(1) = 0 \end{cases} \qquad T' + kT = 0$$

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(b) $X'' + \lambda^2 X = 0$ has solutions:

$$X(x) = A\cos\lambda x + B\sin\lambda x.$$

$$0 = X(0) = A$$

$$X'(x) = B\lambda\cos\lambda x$$

$X'(1)=0 \Rightarrow B = 0$ (trivial)

or $\lambda = 0$ (trivial sol)

or $\cos\lambda = 0 \Rightarrow \lambda = \lambda_n = \frac{2n+1}{2}\pi, \; n = 0, 1, 2, \dots$

(c) The product solutions are

$\sin(\lambda_n x)\exp[-\lambda_n^2 t]$

So by principle of superposition,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x)\exp[-\lambda_n^2 t]}$$

(d) $\underset{\substack{\parallel \\ f(x)}}{u(x,0)} = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x)$

$$b_n = \frac{(\sin(\lambda_n x), f(x))}{(\sin(\lambda_n x), \sin(\lambda_n x))}$$

$$= \boxed{2\int_0^1 \sin(\lambda_n x) f(x)\, dx}$$

(e) RHS is given in sine series form already.

$$b_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \boxed{u(x,t) = \sin\left(\frac{\pi}{2}x\right)\exp\left[-\left(\frac{\pi}{2}\right)^2 t\right]}$$

Problem 3 (30 pts) Consider the 2D Laplace equation below which models the steady state temperature distribution of a square plate where all the sides but the top one ($y = 1$) are dipped in an ice bath,

$$\text{(LAP)} \quad \begin{cases} u_{xx} + u_{yy} = 0, \; 0 < x < 1 \text{ and } 0 < y < 1, \\ u(0, y) = u(1, y) = 0, \; 0 < y < 1, \\ u(x, 0) = 0, \; 0 < x < 1, \\ u(x, 1) = f(x), \; 0 < x < 1. \end{cases}$$

Separation of variables with $u(x, y) = X(x)Y(y)$ gives

$$X'' + kX = 0, \; X(0) = 0, \; X(1) = 0$$
$$Y'' - kY = 0, \; Y(0) = 0.$$

(a) Assuming $k = \mu^2 > 0$, obtain the product solutions to (LAP):
$$u_n(x, y) = B_n \sin(n\pi x) \sinh(n\pi y).$$

(b) Write down the general form of a solution to (LAP) and express B_n in terms of $f(x)$.
(c) Solve (LAP) with $f(x) = 100$.

(a) $k = \mu^2 > 0$

$\Rightarrow X(x) = A \cos \mu x + B \sin \mu x$

$X(0) = 0 \Rightarrow A = 0$

$X(1) = 0 \Rightarrow B \sin(\mu) = 0 \Rightarrow \begin{cases} B = 0 \text{ (trivial)} \\ \text{or} \\ \sin(\mu) = 0 \Rightarrow \mu = \underline{\mu_n = n\pi} \end{cases}$

$\Rightarrow \boxed{X_n(x) = \sin(n\pi x)}$

$Y(y) = A \cosh \mu y + B \sinh \mu y$

$Y(0) = 0 \Rightarrow A \underbrace{\cosh(0)}_{=1} = 0$

$\Rightarrow \boxed{Y_n(y) = B_n \sinh(n\pi y)}$

$\rightarrow \boxed{u_n(x, y) = B_n \sin(n\pi x) \sinh(n\pi y)}$

(b) General solution:

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) \sinh(n\pi y)}$$

(c)

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$$u(x,1) = f(x)$$
$$\| $$
$$\sum_{n=1}^{\infty} \sin(n\pi x)\, B_n \sinh(n\pi)$$

$$B_n \sinh(n\pi) \equiv \text{sine series coeff of } f(x)$$
$$= 2\int_0^1 \sin(n\pi x)\, f(x)\, dx$$

$$\Rightarrow \quad B_n = \frac{2}{\sinh(n\pi)} \int_0^1 \sin(n\pi x)\, dx$$
$$= \frac{2}{\sinh(n\pi)} \left(-\cos(n\pi)\right)\Big|_0^1$$
$$= \boxed{\frac{2}{\sinh(n\pi)} \left(1-(-1)^n\right)}$$

Problem 4 (20 pts) Consider a circular plate with radius 2, with its rim dipped in an ice bath. Its temperature at $t = 0$ is $f(r)$ (i.e. radially symmetric). The temperature distribution $u(r,t)$ of the plate is also radially symmetric and satisfies the 2D Heat equation

$$\text{(2DHEQ)} \qquad \begin{cases} u_t = \Delta u, & \text{for } 0 < r < 2 \text{ and } t > 0 \\ u(r,0) = f(r), & \text{for } 0 < r < 2 \\ u(2,t) = 0, & \text{for } t > 0. \end{cases}$$

Carrying out the method of separation of variables gives that a general solution to (2DHEQ) has the form

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\tfrac{\alpha_n}{2} r) \exp[-\left(\tfrac{\alpha_n}{2}\right)^2 t],$$

with α_n being the n-th positive zero of the first kind zeroth order Bessel function $J_0(r)$.

(a) Use the initial conditions and the orthogonality relations for Bessel functions (see end of this exam) to show that

$$A_n = \frac{1}{2J_1^2(\alpha_n)} \int_0^2 f(r) J_0(\tfrac{\alpha_n}{2} r) r\, dr.$$

(b) Solve (2DHEQ) with $f(r) = 1$. **Hint:** Use the last integration formula in §0.3 to show that:

$$A_n = \frac{2}{\alpha_n J_1(\alpha_n)}.$$

(a) $u(r,0) = f(r) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n}{2} r\right)$

$$\Rightarrow \quad A_n = \frac{\left(f(r), J_0\left(\frac{\alpha_n}{2} r\right)\right)}{\left(J_0\left(\frac{\alpha_n}{2} r\right), J_0\left(\frac{\alpha_n}{2} r\right)\right)}$$

$$= \frac{2}{2^2 J_1^2(\alpha_n)} \int_0^2 f(r) J_0\left(\frac{\alpha_n}{2} r\right) r\,dr$$

C.O.V $r = \frac{2}{\alpha_n} s$

(b) $$A_n = \frac{1}{2J_1^2(\alpha_n)} \int_0^2 1 \times J_0\Big(\underbrace{\frac{\alpha_n}{2} r}_{s}\Big) r\,dr$$

$$\overset{\text{C.O.V}}{=} \frac{1}{2J_1^2(\alpha_n)} \int_0^{\alpha_n} J_0(s) \left(\frac{2}{\alpha_n}\right)^2 s\, ds$$

$$= \frac{2}{\alpha_n^2 J_1^2(\alpha_n)} \int_0^{\alpha_n} s J_0(s)\, ds \overset{\S 0.3}{=} \frac{2}{\alpha_n^2 J_1^2(\alpha_n)} s J_1(s) \Big|_0^{\alpha_n} = \frac{2}{\alpha_n J_1(\alpha_n)}$$

Problem 5 (10 pts) Use the Laplacian in polar coordinates to determine whether the function $f(x,y) = \tan^{-1}(\frac{y}{x})$ satisfies Laplace's equation $\Delta u = 0$.

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

$$f(r,\theta) = \theta \quad \Rightarrow \quad \Delta u = 0.$$