PRACTICE MIDTERM #2 SOLUTIONS

Prob 1

$$\begin{cases} u_t = 3u_{xx} & t>0,\ x\in(0,1) \\ u_x(0,t)=0 & t>0 \\ u(1,t)=0 & t>0 \\ u(x,0)=f(x) & x\in(0,1) \end{cases}$$

(a) Take $u(x,t)=X(x)T(t)$ and plug into PE:

$$XT' = 3X''T$$

$$\frac{T'}{3T} = \frac{X''}{X} = -\lambda^2 \quad (\text{negative const so that } T\to 0 \text{ as } t\to\infty)$$

$$\Rightarrow \boxed{T' - 3\lambda^2 T = 0} \quad \& \quad \boxed{\begin{cases} X'' + \lambda^2 X = 0 \\ X'(0) = X(1) = 0 \end{cases}}$$

$$\Rightarrow X(x) = a\cos\lambda x + b\sin\lambda x$$
$$X'(x) = -a\lambda\sin\lambda x + b\lambda\cos\lambda x$$

Using B.C.:

$$X'(0) = b\lambda = 0 \quad \Rightarrow \quad b=0 \text{ or } \lambda=0$$

if $b=0 \Rightarrow X(x) = a\cos\lambda x$

$$X(1) = a\cos\lambda = 0 \Rightarrow \lambda = \lambda_n = \frac{2n+1}{2}\pi,\ n=0,1,2\ldots$$

if $\lambda = 0$: $X(x) = a$, $X(1) = 0 \Rightarrow a=0$ (trivial sol).

$$\Rightarrow \boxed{\begin{aligned} X_n(x) &= \cos(\lambda_n x) \\ T_n(t) &= a_n\exp[-\lambda_n^2 3t] \end{aligned}}$$

product solutions are:

$$u_n(x,t) = a_n\cos(\lambda_n x)\exp[-3\lambda_n^2 t]$$

$\Rightarrow$ general form is: $u(x,t) = \sum_{n=0}^{\infty} a_n\cos(\lambda_n x)\exp[-3\lambda_n^2 t]$

(b) at $t=0$: $u(x,0) = f(x) = \sum_{n=0}^{\infty} a_n \cos \lambda_n x$

$\Rightarrow (f, \cos \lambda_n x) = a_n (\cos \lambda_n x, \cos \lambda_n x)$ (by $\perp$)

$\Rightarrow \boxed{a_n = \dfrac{(f, \cos \lambda_n x)}{(\cos \lambda_n x, \cos \lambda_n x)} = 2\int_0^1 f(x) \cos \lambda_n x \, dx.}$

(c) With $\perp$ relations we see that:

$$a_n = \begin{cases} 1 & \text{if } n=1 \\ 2 & \text{if } n=3 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \boxed{u(x,t) = \cos\left(\frac{3\pi x}{2}\right) \exp\left[-3\left(\frac{3\pi}{2}\right)^2 t\right] + 2\cos\left(\frac{7\pi}{2}x\right) \exp\left[-3\left(\frac{7\pi}{2}\right)^2 t\right]}$$

Problem 2

(a) Eq. (2) is a homog linear eq. Thus if v solves (P1) and w solves (P2) then $u = v + w$ solves (2).

(b) Plugin $w(x,y) = X(x)Y(y)$ into (2) gives:

$$XY'' + X''Y = 0 \iff \frac{X''}{X} = -\frac{Y''}{Y} = -k$$

$$\begin{cases} X'' + kX = 0 \\ X(0) = X(1) = 0 \end{cases} \quad \text{and} \quad \begin{cases} Y'' - kY = 0 \\ Y(0) = 0 \end{cases}$$

from BC at right & left sides

comes from BC at bottom side.

(c) Solving for X:

$X(x) = a\cos\mu x + b\sin\mu x$

$X(0) = a = 0$

$X(1) = b\sin\mu = 0$

$\Rightarrow \mu = \mu_n = n\pi$

$\Rightarrow X_n(x) = \sin(n\pi x)$

$(n = 1, 2, \dots)$

Solving for Y

$Y(y) = a\cosh\mu_n y + b\sinh\mu_n y$

$Y_n(0) = 0 = a$

$\Rightarrow Y_n(y) = b_n \sinh(\mu_n y)$

$= b_n \sinh(n\pi y)$

$(n = 1, 2, \dots)$

Thus product solutions are:

$$w_n(x,y) = B_n \sin(n\pi x)\sinh(n\pi y)$$

(d) $$w(x,y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sinh(n\pi y)$$

B.C.:

$$w(x,1) = f_2(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sinh n\pi$$

$$\Rightarrow B_n \sinh n\pi = \frac{(f_2, \sin n\pi x)}{(\sin n\pi x, \sin n\pi x)}$$

$$= 2\int_0^1 f_2(x)\sin n\pi x \, dx$$

$$\Rightarrow \boxed{B_n = \frac{2}{\sinh n\pi}\int_0^1 f_2(x)\sin n\pi x \, dx}$$

(e) The general form for a sol to (P1) is:

$$V(x,y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi(1-y)$$

B.C.: $\underset{f_1(x)}{\underset{\parallel}{V(x,0)}} = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi$

$$\Rightarrow \boxed{A_n = \frac{2}{\sinh n\pi} \int_0^1 \sin n\pi x \, f_1(x)\, dx}$$

(f) $f_1(x) = 100, \quad f_2(x) = 100x(1-x)$

$$A_n = \frac{2}{\sinh n\pi} \int_0^1 \sin n\pi x \,(100)\, dx$$

$$= \frac{200}{\sinh n\pi} \frac{(1-(-1)^n)}{n\pi}$$

$$B_n = \frac{2}{\sinh n\pi} \int_0^1 \sin n\pi x \left(100x(1-x)\right) dx$$

$$= \frac{400}{\sinh n\pi} \frac{((-1)^n - 1)}{\pi^3 n^3}$$

$\Rightarrow$ sol. to (2) is:

$$u(x,y) = V(x,y) + w(x,y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi(1-y)$$

$$+ \sum_{n=1}^{\infty} B_n \sin n\pi x \sinh n\pi y$$

Problem 3

(a) Using B.C.:

$$u(r,0) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_{0n}}{\ell} r\right) = f(r)$$

thus:

$$A_n = \frac{\left(f(r), J_0\left(\frac{\alpha_{0n}}{2} r\right)\right)}{\left(J_0\left(\frac{\alpha_{0n}}{2} r\right), J_0\left(\frac{\alpha_{0n}}{2} r\right)\right)}$$

$$= \left(\frac{4}{2} J_1^2(\alpha_{0n})\right)^{-1} \int_0^1 f(r) J_0\left(\frac{\alpha_{0n}}{2} r\right) r\,dr$$

$$= \frac{1}{2 J_1^2(\alpha_{0n})} \int_0^1 f(r) J_0\left(\frac{\alpha_{0n}}{2} r\right) dr.$$

(b) From §0.3:

$$\int_0^2 (4-r^2) J_0\left(\frac{\alpha_{0n}}{2} r\right) r\,dr = \frac{2 \cdot 2^4}{\alpha_n^2} J_2(\alpha_n)$$

$$\Rightarrow A_n = \frac{1}{2 J_1^2(\alpha_{0n})} \frac{2 \cdot 2^4}{\alpha_{0n}^2} J_2(\alpha_n) = \boxed{\frac{16 J_2(\alpha_{0n})}{\alpha_{0n}^2 J_1^2(\alpha_{0n})}}$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{16 J_2(\alpha_{0n})}{\alpha_{0n}^2 J_1^2(\alpha_{0n})} J_0\left(\frac{\alpha_n}{2} r\right) \cos\left(\frac{\alpha_n}{2} t\right)}$$

Problem 4: The Laplacian in polar coord is:

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

$$f(x,y) = \ln(x^2+y^2) = \ln(r^2)$$
$$= 2\ln(r)$$

$$\left.\begin{array}{l} f_r = \frac{2}{r} \\ f_{rr} = -\frac{2}{r^2} \end{array}\right\} \Rightarrow f_{rr} + \frac{1}{r} f_r = -\frac{2}{r^2} + \frac{1}{r}\frac{2}{r} = 0$$

$$f_{\theta\theta} = 0$$

$\Rightarrow f$ satisfies Laplace eq.