## MATH 3150-4, PRACTICE MIDTERM EXAM 2 SPRING 2012

## **Total points:** 100/100.

**Problem 1 (30 pts)** The goal of this problem is to solve the Heat Equation with *mixed* boundary conditions

(1) 
$$\begin{cases} u_t = 3u_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u_x(0,t) = 0 & \text{for } t > 0 \\ u(1,t) = 0 & \text{for } t > 0 \\ u(x,0) = f(x) & \text{for } 0 < x < 1 \end{cases}$$

(a) Use separation of variables to show that a general solution to (1) is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos(\lambda_n x) \exp[-3\lambda_n^2 t], \text{ where } \lambda_n = \frac{2n+1}{2}\pi.$$

(b) Consider the inner product  $(u, v) = \int_0^1 u(x)v(x)dx$ . Given the orthogonality relations valid for  $n = 0, 1, 2, \ldots$  and  $m = 0, 1, 2, \ldots$ 

$$(\cos(\lambda_n x), \cos(\lambda_m x)) = \begin{cases} \frac{1}{2} & \text{if } n = m\\ 0 & \text{if } n \neq m, \end{cases}$$

show that

$$a_n = 2 \int_0^1 \cos(\lambda_n x) f(x) dx$$
, for  $n = 0, 1, 2, ...$ 

(c) Solve problem (1) with  $f(x) = \cos(3\pi x/2) + 2\cos(7\pi x/2)$ .

**Problem 2 (30 pts)** Consider the 2D Laplace equation below, which models the steady state temperature distribution of a square plate where the right and left sides are kept in an ice bath and the bottom and top sides have prescribed temperatures  $f_1(x)$  and  $f_2(x)$  respectively.

(2) 
$$\begin{cases} u_{xx} + u_{yy} = 0, \text{ for } 0 < x < 1 \text{ and } 0 < y < 1 \\ u(0, y) = u(1, y) = 0, \text{ for } 0 < y < 1 \\ u(x, 0) = f_1(x), \text{ for } 0 < x < 1 \\ u(x, 1) = f_2(x), \text{ for } 0 < x < 1. \end{cases}$$

(a) Explain why it is possible to decompose (2) into the two subproblems below (the x and y below are implicitly in (0, 1)).

$$(P1) \begin{cases} v_{xx} + v_{yy} = 0, \\ v(0, y) = v(1, y) = 0, \\ v(x, 0) = f_1(x), \\ v(x, 1) = 0 \end{cases} (P2) \begin{cases} w_{xx} + w_{yy} = 0, \\ w(0, y) = w(1, y) = 0, \\ w(x, 0) = 0, \\ w(x, 1) = f_2(x) \end{cases}$$

(b) Show that if we assume that the solution to (P2) is w(x, y) = X(x)Y(y), then separation of variables gives

$$X'' + kX = 0, \quad X(0) = 0, \quad X(1) = 0$$
$$Y'' - kY = 0, \quad Y(0) = 0$$

(c) Assuming  $k = \mu^2 > 0$ , obtain the product solutions to (P2)

 $w_n(x,y) = B_n \sin(n\pi x) \sinh(n\pi y)$ 

- (d) Write down the general form of a solution to (P2), and use the formulas at the end of the exam to express  $B_n$  in terms of  $f_2(x)$ .
- (e) In a similar way it is possible to obtain the product solutions to (P1),

 $v_n(x,y) = A_n \sin(n\pi x) \sinh(n\pi(1-y)).$ 

Write down the general form of a solution to (P1) and give an expression for  $A_n$  in terms of  $f_1(x)$ .

(f) Solve (2) with  $f_1(x) = 100$  and  $f_2(x) = 100x(1-x)$ . You may use the identity below (valid for n = 1, 2, ...):

$$\int_0^1 x(1-x)\sin(n\pi x)dx = \frac{2((-1)^n - 1)}{\pi^3 n^3}.$$

**Problem 3 (30 pts)** Consider a circular membrane of radius 2 with radially symmetric initial shape f(r) and zero initial velocity. Then the displacement from equilibrium satisfies the 2D Wave equation

(3) 
$$\begin{cases} u_{tt} = \Delta u, & \text{for } 0 < r < 2 \text{ and } t > 0 \\ u(r,0) = f(r), & \text{for } 0 < r < 2 \\ u_t(r,0) = 0, & \text{for } 0 < r < 2 \\ u(2,t) = 0, & \text{for } t > 0. \end{cases}$$

Carrying out the method of separation of variables gives that a general solution to (3) has the form

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\frac{\alpha_n}{2}r) \cos(\frac{\alpha_n}{2}t),$$

with  $\alpha_n$  being the *n*-th positive zero of the first kind zeroth order Bessel function  $J_0(r)$ .

(a) Use the initial conditions and the orthogonality relations for Bessel functions (see end of this exam) to show that

$$A_n = \frac{1}{2J_1^2(\alpha_n)} \int_0^2 f(r) J_0(\frac{\alpha_n}{2}r) r \, dr.$$

(b) Solve (3) with  $f(r) = 4 - r^2$ . Hint: Use the last integration formula in §0.3. **Problem 4 (10 pts)** Determine whether the function  $f(x, y) = \ln(x^2 + y^2)$  satisfies Laplace's equation  $\Delta u = 0$ .

## Some useful formulas

0.1. Orthogonality relations for sine series. With the inner product  $(u, v) = \int_0^a u(x)v(x)dx$ , we have for all m, n non-zero integers,

$$\left(\sin\left(\frac{m\pi}{a}x\right),\sin\left(\frac{n\pi}{a}x\right)\right) = \begin{cases} \frac{2}{a} & \text{if } m = n\\ 0 & \text{if } m \neq n \end{cases}$$

## 0.2. Hyperbolic trigonometry.

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$
$$(\cosh x)' = \sinh x, \quad (\sinh x)' = \cosh x$$
$$\cosh^2 x - \sinh^2 y = 1, \qquad \qquad \sinh 0 = 0$$

0.3. Bessel functions. The following identities are valid for  $p \ge 0$  and  $n = 0, 1, \ldots$ 

$$\int J_1(r)dr = -J_0(r) + C \quad \text{and} \quad \int r^{p+1} J_p(r)dr = r^{p+1} J_{p+1}(r) + C$$

0.4. Orthogonality relations for Bessel functions. Let a > 0 and  $m \ge 0$  be fixed. Denote with  $\alpha_{mn}$  the *n*-th positive zero of the Bessel function of the first kind of order *m*. With the inner product

$$(u,v) = \int_0^a u(r)v(r)r\,dr$$

we have for all j, k non-zero integers,

$$\left(J_m(\frac{\alpha_{mj}}{a}r), J_m(\frac{\alpha_{mk}}{a}r)\right) = \begin{cases} \frac{a^2}{2}J_{m+1}^2(\alpha_{mj}) & \text{if } j = k\\ 0 & \text{if } j \neq k. \end{cases}$$