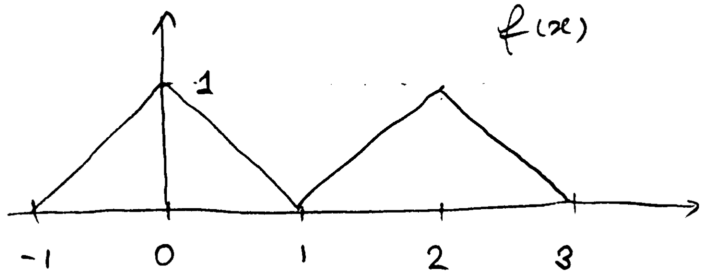
MATH 3150-4 MIDTERM 1

SOLUTIONS

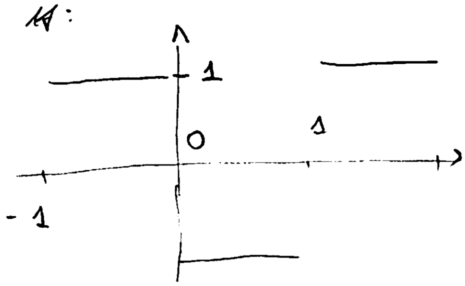
Problem 1 2-periodic function $f(x) = \begin{cases} 1+x & \text{if } -1 \le x \le 0 \\ 1-x & \text{if } 0 < x \le 1 \end{cases}$

(a)



(b) f is continuous and pcws cont.

f' is:



also pcws cont $\Rightarrow$ f is pcws smooth.

(c) f is even: $f(-x) = f(x)$

(d) Fourier series of $f(x)$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x.$$

Since f is odd: $\boxed{b_n = 0,}\ n = 1, 2, \ldots$

$$a_0 = \frac{(f, 1)}{(1, 1)} = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\Bigg[\underbrace{\int_{-1}^{0}(1+x)\,dx}_{=\frac{1}{2}} + \underbrace{\int_{0}^{1}(1-x)\,dx}_{=\frac{1}{2}}\Bigg]$$

$$= \frac{1}{2}$$

$$a_n = \frac{(f, \cos n\pi x)}{(\cos n\pi x, \cos n\pi x)} = \int_{-1}^{1} f(x) \cos n\pi x \, dx$$

$$= \int_{-1}^{0} (1+x) \cos n\pi x \, dx + \int_{0}^{1} (1-x) \cos n\pi x \, dx$$

$$\overset{IBP}{=} (1+x) \frac{\sin n\pi x}{n\pi}\Big|_{-1}^{0} - \int_{-1}^{0} \frac{\sin n\pi x}{n\pi} dx$$

$$+ (1-x) \frac{\sin n\pi x}{n\pi}\Big|_{0}^{1} + \int_{0}^{1} \frac{\sin n\pi x}{n\pi} dx$$

$$= + \frac{\cos n\pi x}{(n\pi)^2}\Big|_{-1}^{0} - \frac{\cos n\pi x}{(n\pi)^2}\Big|_{0}^{1}$$

$$= \frac{1}{(n\pi)^2} \left(2 - 2(-1)^n\right)$$

$$\Rightarrow f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} \left(1 - (-1)^n\right) \cos n\pi x$$

$$= \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2} \cos (2k+1)\pi x$$

Problem 2

(a) $u_{xx} + u_{xy} = 2u$: linear, homog., 2nd order

(b) $u_x + x\, u_{xy} = 2$ linear, non-homog, 2nd order

(c) $u_{xx} - u_t = \sin(x+t)$ linear, non-homog, 2nd order

(d) $u_{xx} + u^2 = u_t$ non-linear, 2nd order.

Problem 3

(a) $\cos \pi x$ is 2-periodic: $\cos \pi(x+2) = \cos(\pi x + 2\pi) = \cos(\pi x)$

(b) $e^{\sin(12x)}$ is $\frac{\pi}{6}$ periodic: $e^{\sin(12(x+\pi/6))} = e^{\sin(12x + 2\pi)} = e^{\sin(12x)}$

(c) $\cos(\pi x) + \sin(\pi x/2)$ is 4 periodic: $\cos(\pi(x+4)) + \sin(\pi(x+4)/2)$

$$= \cos(\pi x + 4\pi) + \sin(\pi x/2 + 2\pi)$$
$$= \cos(\pi x) + \sin(\pi x/2).$$

(d) $\cos(\pi x) \sin(\pi x)$ is 2-period:

$$\cos(\pi(x+2)) \sin(\pi(x+2)) = \cos(\pi x + 2\pi) \sin(\pi x + 2\pi) = \cos(\pi x) \sin(\pi x)$$

Problem 4 $u(x,t) = f(x+ct) + f(x-ct)$ satisfies WEQ since:

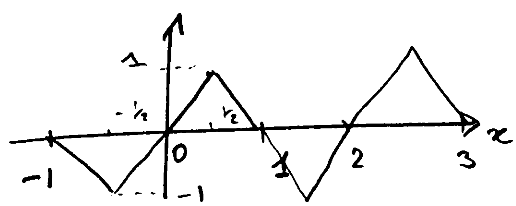
$$u_t = c f'(x+ct) - c f'(x-ct)$$
$$u_{tt} = c^2 f''(x+ct) + c^2 f''(x-ct)$$
$$u_x = f'(x+ct) + f'(x-ct)$$
$$u_{xx} = f''(x+ct) + f''(x-ct)$$

$$\Rightarrow \quad \boxed{u_{tt} = c^2 u_{xx}}$$

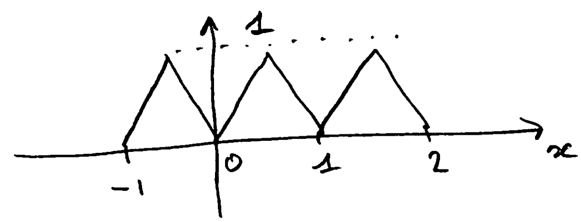
Problem 5 f is defined on $[0,1]$ by:

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

(a) odd 2-periodic extension



even 2-periodic extension



(b) Sine series of $f(x)$:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$b_n = \frac{(f(x), \sin(n\pi x))}{(\sin(n\pi x), \sin(n\pi x))} \qquad \text{where } (u,v) = \int_0^1 u(x)v(x)\,dx$$

$$= 2\int_0^1 f(x) \sin(n\pi x)\,dx$$

$$= 2\int_0^{\frac{1}{2}} x \sin(n\pi x)\,dx + 2\int_{\frac{1}{2}}^1 (1-x)\sin(n\pi x)\,dx$$

$$\overset{IBP}{=} 2x\left(-\frac{\cos n\pi x}{n\pi}\right)\Bigg|_0^{\frac{1}{2}} + 2\int_0^{\frac{1}{2}} \frac{\cos n\pi x}{n\pi}\,dx$$
$$+ 2(1-x)\left(-\frac{\cos n\pi x}{n\pi}\right)\Bigg|_{\frac{1}{2}}^1 - 2\int_{1/2}^1 \frac{\cos n\pi x}{n\pi}\,dx$$

$$b_n = 2\,\frac{\sin(n\pi x)}{(n\pi)^2}\Bigg|_0^{\frac{1}{2}} - 2\,\frac{\sin(n\pi x)}{(n\pi)^2}\Bigg|_{1/2}^{1}$$

$$= \frac{4\sin(n\pi/2)}{(n\pi)^2}$$

n	$\sin(n\pi/2)$
1	1
2	0
3	-1
4	0
5	1
6	0
⋮	⋮

$$\Rightarrow \boxed{\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{(n\pi)^2}\sin(n\pi x) \\ &= \sum_{k=0}^{\infty} \frac{4(-1)^k}{((2k+1)^2\pi^2)}\sin((2k+1)\pi x) \end{aligned}}$$

(c) The solution to:

$$\begin{cases} u_{tt} = 3u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$$

is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{(n\pi)^2} \sin(n\pi x) \cos(\sqrt{3}\, n\pi t)}$$

(d) The solution to:

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is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{(n\pi)^2} \sin(n\pi x) \frac{1}{\sqrt{3}\, n\pi} \sin(\sqrt{3}\, n\pi t)}$$