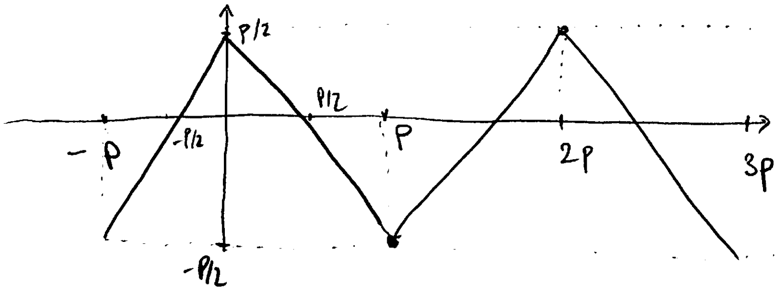
MATH 3150 Practice Midterm Exam Solutions

Prob 1

(a) Here is a sketch of $f(x)$ for $x \in [-p, 3p]$.

$$f(x) = \begin{cases} -(x - p/2) & \text{if } 0 \le x \le p \\ x + p/2 & \text{if } -p \le x \le 0 \end{cases}$$



(b) $f(x)$ is continuous $\Rightarrow$ pcws continuous

$$f'(x) = \begin{cases} -1 & 0 < x < p \\ +1 & -p < x < 0 \end{cases} \quad , \ 2p\text{- periodic}$$

$\Rightarrow f'$ is pcws cont. $\Rightarrow f$ is pcws smooth

(c) Since $f(x)$ is an <u>even</u> function: $(f(-x) = f(x))$

$$b_n = \frac{(f, \sin\frac{n\pi}{p}x)}{(\sin\frac{n\pi}{p}x, \sin\frac{n\pi}{p}x)} = \frac{1}{p}\int_{-p}^{p} \underbrace{\underbrace{f(x)}_{\text{EVEN}} \underbrace{\sin\frac{n\pi}{p}x}_{\text{ODD}}}_{\text{ODD}}\, dx = 0$$

$$a_n = -\frac{2}{p}\int_0^p \underset{\downarrow}{\left(x - \frac{p}{2}\right)} \underset{\uparrow}{\cos\frac{n\pi}{p}x}\, dx$$

$$\overset{IBP}{=} \overbrace{-\frac{2}{p}(x - p/2)\frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_0^p}^{=0} + \frac{2}{p}\frac{p}{n\pi}\int_0^p \sin\frac{n\pi}{p}x\, dx$$

$$= -\frac{2}{n\pi}\left(\frac{p}{n\pi}\right)\cos\frac{n\pi}{p}x\Big|_0^p = \frac{-2p}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$= \begin{cases} \frac{4p}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$a_0 = \frac{1}{p}\int_0^p f(x)\,dx = 0$ so formula works for n = 0.

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$$

$$= \sum_{k=0}^{\infty} \frac{4p}{(2k+1)^2\pi^2} \cos \frac{(2k+1)\pi}{p} x$$

* * *

Prob 2

(a) fourth order, linear and homog

(b) second order, non-linear

(c) second order, linear non-homog

(d) second order, linear, homog.

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Prob 3

(a) $\sin(2(x+\pi)) = \sin(2x + 2\pi) = \sin 2x$

period $= \pi$

(b) $\underbrace{\cos\left(\frac{x}{2}\right)}_{\text{period} = 4\pi,\, 8\pi, \ldots} + 3\underbrace{\sin 2x}_{\text{period} = \pi, 2\pi, 3\pi, \ldots}$

$\Rightarrow$ common period is 4π

Check: $\cos\left(\frac{x+4\pi}{2}\right) + 3\sin(2(x+4\pi))$

$= \cos\left(\frac{x}{2}+2\pi\right) + 3\sin(2x+8\pi)$

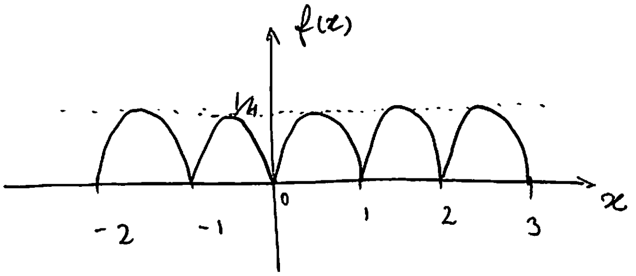
$= \cos\left(\frac{x}{2}\right) + 3\sin(2x)$

(c) $\frac{1}{2+\sin(x+2\pi)} = \frac{1}{2+\sin(x)}$ $\Rightarrow$ 2π-periodic

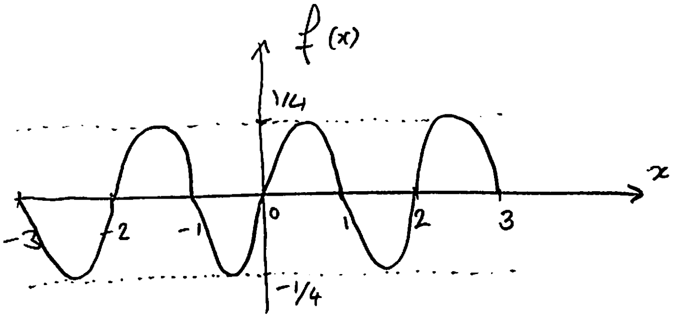
Problem 4 $f(x) = x(1-x)$

(a)

EVEN 2-per. ext.



ODD 2-per ext.



(b) Sine series:

$$b_n = \frac{2}{1}\int_0^1 \underset{\downarrow}{x(1-x)}\ \underset{\uparrow}{\sin(n\pi x)}\,dx$$

$$\overset{IBP}{=} \underbrace{-\frac{2}{n\pi}x(1-x)\cos(n\pi x)\Big|_0^1}_{=0} + \frac{2}{n\pi}\int_0^1 \underset{\downarrow}{(1-2x)}\ \overset{\uparrow}{\cos(n\pi x)}\,dx$$

$$\overset{IBP}{=} \underbrace{\frac{2}{(n\pi)^2}(1-2x)\sin n\pi x\Big|_0^1}_{=0} - \frac{2}{(n\pi)^2}\int_0^1(-2)\sin n\pi x\,dx$$

$$= \frac{4}{(n\pi)^2}\,\frac{-\cos(n\pi x)}{n\pi}\Big|_0^1$$

$$= \frac{4}{(n\pi)^3}\left(1-(-1)^n\right) = \begin{cases} \frac{8}{(n\pi)^3} & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

$$\Rightarrow \boxed{f(x) = \sum_{k=0}^{\infty}\frac{8}{(2k+1)^3\pi^3}\sin\left((2k+1)\pi x\right)}$$

For cosine series :

$$a_0 = \int_0^1 x(1-x)\,dx = \frac{x^2}{2}-\frac{x^3}{3}\Big|_0^1 = \frac{1}{2}-\frac{1}{3} = \boxed{\frac{1}{6}}$$

$$a_n = 2\int_0^1 \underset{\downarrow}{x(1-x)}\,\underset{\uparrow}{\cos(n\pi x)}\,dx$$

$$\overset{IBP}{=} \underbrace{\frac{2}{n\pi}x(1-x)\sin(n\pi x)\Big|_0^1}_{=0} - \frac{2}{n\pi}\int_0^1 \underset{\downarrow}{(1-2x)}\,\underset{\uparrow}{\sin(n\pi x)}$$

$$\overset{IBP}{=} \frac{2}{(n\pi)^2}(1-2x)\cos(n\pi x)\Big|_0^1 - \frac{2}{(n\pi)^2}\int_0^1 (-2)\cos(n\pi x)\,dx$$

$$= \frac{2}{(n\pi)^2}\left(-(-1)^n-1\right) + \underbrace{\frac{4}{(n\pi)^2}\,\frac{\sin n\pi x}{n\pi}\Big|_0^1}_{=0}$$

$$= \begin{cases} -\dfrac{4}{(n\pi)^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{6}+\sum_{k=1}^{\infty} -\frac{4}{(2k)^2 n^2}\cos((2k)\pi x)}$$

(c) We need to solve: $\begin{cases} u_{tt} = 2u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$ $\Rightarrow b_n^* = 0.$

should be sqrt(2)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \cos 2n\pi t$$

$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \Rightarrow$ bn are same as sine series of f(x).

$$\Rightarrow \boxed{u(x,t) = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^3\pi^3}\sin((2k+1)\pi x)\cos(2(2k+1)\pi t)}$$

should be sqrt(2)

(d) We need to solve.

$$\begin{cases} u_{tt} = 2u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = f(x) \end{cases} \Rightarrow b_n = 0.$$

should be sqrt(2)

$$u(x,t) = \sum_{n=1}^{\infty} b_n^* \sin(n\pi x) \sin(2n\pi t)$$

should be sqrt(2)

$$u_t(x,t) = \sum_{n=1}^{\infty} (2n\pi) b_n^* \sin(n\pi x) \cos(2n\pi t)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} (2n\pi)\, b_n^* \sin(n\pi x) = f(x)$$

should be sqrt(2)

$$\Rightarrow \quad (2n\pi) b_n^* = b_n = n\text{-th sine series coeff of } f(x)$$

should be 8/sqrt(2)

$$\Rightarrow \quad b_n^* = \begin{cases} \frac{4}{(n\pi)^4} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\Rightarrow \quad \boxed{u(x,t) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)^4 \pi^4} \sin((2k+1)\pi x) \sin(2(2k+1)\pi t)}$$

should be 8/sqrt(2)

should be sqrt(2)