Review of numerical methods for first order differential equations Math 2280-2

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We review different numerical methods for getting an approximate solution to the initial value problem,

$$\begin{aligned} \frac{dy}{dx} &= f(x,y) \\ y(x_0) &= y_0, \end{aligned} \tag{1}$$

on some $x-$interval $[x_0, x_n]$. First let us fill the interval $[x_0, x_n]$ with $n+1$ points

$$x_i = x_0 + ih, \quad \text{for } i = 0 \ldots n.$$

Here $h = 1/(x_n - x_1)$ is the "step size". The basic principle of these methods is to somehow approximate $y_{i+1} \approx y(x_{i+1})$ based on previous iterates. One way to achieve this is by first integrating the DE in (1) between x_i and x_{i+1},

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} f(x, y(x))dx \tag{2}$$

and then using numerical integration or quadrature rules for approximating the integral in (2).

1 Euler's method

The simplest method is to use the following quadrature rule (which could be called "left point" rule) to approximate the integral (2):

$$\int_{x_i}^{x_{i+1}} f(x, y(x))dx \approx hf(x_i, y(x_i))$$

Note that the length of the interval over which we integrate is $h = x_{i+1} - x_i$. In pseudocode this gives:

Euler's method
for $i = 0 \ldots n-1$
    $k \leftarrow f(x_i, y_i)$
    $y_{i+1} \leftarrow y_i + hk$
end for

Work: 1 function evaluation/iteration
Accuracy Assuming the solution $y \in C^2$, this is a *first order method*, meaning that there is some $C > 0$ such that

$$|y(x_n) - y_n| < Ch.$$

2 Improved Euler's method

Now if we use the "trapezoidal rule" to approximate the integral (2) we get:

$$\int_{x_i}^{x_{i+1}} f(x, y(x))dx \approx \frac{h}{2}(f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))).$$

The only problem is that this approximation involves $y(x_{i+1})$ which is what we want to compute! Improved Euler's method uses Euler's approximation to *predict* the value of y_{i+1}, that is: $y_{i+1} \approx y_i + hf(x_i, y_i)$. Then this *corrected* value of the slope is used in the update. Such methods are called "predictor-corrector" methods. In pseudocode we would get,

Improved Euler's method
for $i = 0 \ldots n-1$
    $k_1 \leftarrow f(x_i, y_i)$
    $k_2 \leftarrow f(x_i, y_i + hk_1)$
    $k \leftarrow (k_1 + k_2)/2$
    $y_{i+1} \leftarrow y_i + hk$
end for

Work: 2 function evaluations/iteration
Accuracy: Assuming $f \in C^3$, improved Euler is a *second order method*, i.e. there is some $C > 0$ such that

$$|y(x_n) - y_n| < Ch^2.$$

3 Runge-Kutta

There are several Runge-Kutta methods, but the classical one is the *fourth order* Runge-Kutta method or RK4. This is a popular method because of its simplicity and accuracy. It can be motivated by approximating the integral (2) with Simpson's rule:

$$\int_{x_i}^{x_{i+1}} f(x, y(x))dx \approx \frac{h}{6}(f_i + 4f_{i+1/2} + f_{i+1}),$$

where we have written in short $f_i = f(x_i, y(x_i))$ and $x_{i+1/2} = x_i + h/2$ is the midpoint of the interval $[x_i, x_{i+1}]$. We face the same problem, both $f_{i+1/2}$ and f_{i+1} are not known to us because they involve $y_{i+1/2}$ and y_{i+1}. Runge-Kutta makes clever approximations to these quantities:

1. The value of the slope at the midpoint $x_{i+1/2}$ is $f_{i+1/2}$, and it is approximated in two ways

   (a) Let $k_1 = f(x_i, y_i)$. The first approximations uses Euler's method up to the midpoint: $y_{i+1/2} \approx y_i + (h/2)k_1$. So

   $$f_{i+1/2} \approx k_2 = f(x_{i+1/2}, y_i + (h/2)k_1).$$

   (b) The second approximation uses the first:

   $$f_{i+1/2} \approx k_3 = f(x_{i+1/2}, y_i + (h/2)k_2).$$

2. This improved value of the slope at the midpoint is used to estimate the value at the endpoint x_{i+1}:

   $$f_{i+1} \approx k_4 = f(x_{i+1}, y_i + hk_3).$$

Runge-Kutta method RK4

for $i = 0 \dots n-1$

$k_1 \leftarrow f(x_i, y_i)$

$k_2 \leftarrow f(x_{i+1/2}, y_i + (h/2)k_1)$

$k_3 \leftarrow f(x_{i+1/2}, y_i + (h/2)k_2)$

$k_4 \leftarrow f(x_{i+1}, y_i + hk_3)$

$k \leftarrow (k_1 + 2k_2 + 2k_3 + k_4)/6$

$y_{i+1} \leftarrow y_i + hk$

end for

Work: 4 function evaluations/iteration

Accuracy: Assuming $f \in C^5$, RK4 is a *fourth order method*, i.e. there is some $C > 0$ such that

$$|y(x_n) - y_n| < Ch^4.$$