We review different numerical methods for getting an approximate solution to the initial value problem,

\[
\frac{dy}{dx} = f(x, y) \quad (1)
\]

\[y(x_0) = y_0,\]

on some \( x \)-interval \([x_0, x_n]\). First let us fill the interval \([x_0, x_n]\) with \(n + 1\) points

\[x_i = x_0 + ih, \quad \text{for } i = 0 \ldots n.\]

Here \( h = 1/(x_n - x_1) \) is the “step size”. The basic principle of these methods is to somehow approximate \(y_{i+1} \approx y(x_{i+1})\) based on previous iterates. One way to achieve this is by first integrating the DE in (1) between \(x_i\) and \(x_{i+1}\),

\[y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} f(x, y(x))dx \quad (2)\]

and then using numerical integration or quadrature rules for approximating the integral in (2).
1 Euler’s method

The simplest method is to use the following quadrature rule (which could be called “left point” rule) to approximate the integral (2):

\[ \int_{x_i}^{x_{i+1}} f(x, y(x)) \, dx \approx hf(x_i, y(x_i)) \]

Note that the length of the interval over which we integrate is \( h = x_{i+1} - x_i \). In pseudocode this gives:

**Euler’s method**

for \( i = 0 \ldots n - 1 \)

\[ k \leftarrow f(x_i, y_i) \]

\[ y_{i+1} \leftarrow y_i + hk \]

end for

**Work:** 1 function evaluation/iteration

**Accuracy** Assuming the solution \( y \in C^2 \), this is a first order method, meaning that there is some \( C > 0 \) such that

\[ |y(x_n) - y_n| < Ch. \]
2 Improved Euler’s method

Now if we use the “trapezoidal rule” to approximate the integral (2) we get:

\[ \int_{x_i}^{x_{i+1}} f(x, y(x)) \, dx \approx \frac{h}{2} \left( f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1})) \right). \]

The only problem is that this approximation involves \( y(x_{i+1}) \) which is what we want to compute! Improved Euler’s method uses Euler’s approximation to predict the value of \( y_{i+1} \), that is: \( y_{i+1} \approx y_i + hf(x_i, y_i) \). Then this corrected value of the slope is used in the update. Such methods are called “predictor-corrector” methods. In pseudocode we would get,

**Improved Euler’s method**

```plaintext
for i = 0 \ldots n - 1
    k_1 \leftarrow f(x_i, y_i)
    k_2 \leftarrow f(x_i, y_i + hk_1)
    k \leftarrow (k_1 + k_2)/2
    y_{i+1} \leftarrow y_i + hk
end for
```

**Work:** 2 function evaluations/iteration

**Accuracy:** Assuming \( f \in C^3 \), improved Euler is a second order method, i.e. there is some \( C > 0 \) such that

\[ |y(x_n) - y_n| < Ch^2. \]
3 Runge-Kutta

There are several Runge-Kutta methods, but the classical one is the fourth order Runge-Kutta method or RK4. This is a popular method because of its simplicity and accuracy. It can be motivated by approximating the integral (2) with Simpson’s rule:

$$\int_{x_i}^{x_{i+1}} f(x, y(x)) dx \approx \frac{h}{6} (f_i + 4f_{i+1/2} + f_{i+1}),$$

where we have written in short $f_i = f(x_i, y(x_i))$ and $x_{i+1/2} = x_i + h/2$ is the mid-point of the interval $[x_i, x_{i+1}]$. We face the same problem, both $f_{i+1/2}$ and $f_{i+1}$ are not known to us because they involve $y_{i+1/2}$ and $y_{i+1}$. Runge-Kutta makes clever approximations to these quantities:

1. The value of the slope at the midpoint $x_{i+1/2}$ is $f_{i+1/2}$, and it is approximated in two ways
   
   (a) Let $k_1 = f(x_i, y_i)$. The first approximations uses Euler’s method up to the midpoint: $y_{i+1/2} \approx y_i + (h/2)k_1$. So
   
   $$f_{i+1/2} \approx k_2 = f(x_{i+1/2}, y_i + (h/2)k_1).$$
   
   (b) The second approximation uses the first:
   
   $$f_{i+1/2} \approx k_3 = f(x_{i+1/2}, y_i + (h/2)k_2).$$

2. This improved value of the slope at the midpoint is used to estimate the value at the endpoint $x_{i+1}$:

$$f_{i+1} \approx k_4 = f(x_{i+1}, y_i + hk_3).$$

Runge-Kutta method RK4

for $i = 0 \ldots n - 1$

$$k_1 \leftarrow f(x_i, y_i)$$

$$k_2 \leftarrow f(x_{i+1/2}, y_i + (h/2)k_1)$$

$$k_3 \leftarrow f(x_{i+1/2}, y_i + (h/2)k_2)$$

$$k_4 \leftarrow f(x_{i+1}, y_i + hk_3)$$

$$k \leftarrow (k_1 + 2k_2 + 2k_3 + k_4)/6$$

$$y_{i+1} \leftarrow y_i + hk$$

end for

Work: 4 function evaluations/iteration

Accuracy: Assuming $f \in C^5$, RK4 is a fourth order method, i.e. there is some $C > 0$ such that

$$|y(x_n) - y_n| < C h^4.$$